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Fisheries management and intergenerational equity

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Abstract

We adapt the classical Schaefer model of fisheries management to take into account intergenerational equity, in the line of Sumaila ([18]) and Sumaila and Walters ([19]). The resulting discount rate then is non-constant, and the planner's preferences are time inconsistent, so that optimal solutions are not implementable. In the line of Ekeland and Lazrak ([6], [7]) we define Markov subgame perfect equilibria of the underlying sequential game. We characterize equilibrium strategies by a simple relation, and we reach a robust conclusion, namely that, to take into account intergenerational equity, the rate of time preference, δ, should be replace by $\delta - n$, where n is the rate of growth of the human population.

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1 Intergenerational equity.

It is generally agreed that problems which are costly to solve in the short term but whose solution will yield benefits in the long term are difficult to fix because the short term costs loom large even if the long term benefits are much larger in absolute terms. The main reason for this is that in comparing the present values of policy alternatives, we discount net benefits that will accrue in the future and compare them to net benefits that can be achieved today [9]. Because of the exponential growth of compounded interests, the difficulty becomes greater as the future benefits are more distant. In the case of the environmental and resource problems affecting humanity today, such as greenhouse gas emissions and climate change [17], or overfishing of the world's ocean resources [12], it becomes almost overpowering. The Stern review, for instance, states that the course of the next fifty years is already set, so that the effects of any policy will be felt in the range of 50 to 200 years from now. In the following table, we give the present value of $ 1,000,000 at 50, 100, and 200 years, discounted at 10%, 4.6% and 1.4%, which is the value that the Stern Review took:

	50 ys	100 ys	200 ys
10%	8,519	73	< 0.00
4.6%	105,540	11,140	124
1,4%	499,000	249,000	62,000

This clearly illustrates the importance of discounting. But there exists no financial assets in that range, and therefore no market rates: the longest US Treasury bond matures in 30 years. So one has to take a normative approach. Consider the benchmark case when there is a single good x in the economy, and society consists of a single, infinitely-lived consumer, with utility function $u(x)$, and with a constant rate of time preference $\delta > 0$. The welfare function at time 0 then is:

$$\int_0^\infty e^{-\delta t} u(x(t))\, dt$$

If the utility function is of the form $u(x) = \frac{1}{1-\theta} x^{1-\theta}$ (constant absolute risk aversion), the equilibrium interest rate then is

$$r = \delta + \eta g$$

where g is the growth rate of consumption (see [1] , [2], [15]). This has to be corrected to take into account uncertainty on the growth rate, as well as the existence of environmental goods beside industrial goods: we refer to [10] and to [13] for a survey. In this paper, our concern will be on intergenerational equity, in a framework where $u(x)$ is a linear function of x, so that $\theta = 0$ and $r = \delta$.

The classical interpretation of the discount rate δ is the rate of time preference of the present generation: my discount rate if δ if I am indifferent between receiving one unit of utility today and $e^{-\delta t}$ units at time $t > 0$. This approach clearly fails for the very long term: I will not be around to benefit from any changes that will occur 50 to 200 years from now. In other words, when evaluating environmental policies, one has to take into account the fact that the benefits will not accrue to the present generation, but to others, so that the discount rate no longer can be understood as a pure rate of time preference, but also has the character of a Pareto weight: as pointed out in [18], using the 'discounting clock' of the current generation inflates cost to the current generation and diminishes the benefits to future generations. To deal with this problem, Sumaila [18] introduced the concept of intergenerational discounting, which was further developed by Sumaila and Walters [19], and which takes into account, not only the pure rate of time preference of the present generation, but also the Pareto weight which is attributed to future generations.

We refer to [18] and [19] for the model and the derivation of the discount rate in the case when the time is discrete, $t = 0, 1, ...$ However, many important models in economic theory are in continous time, for instance the classical Ramsey model of economic growth (see [1] , [2], [15]), or, closer to our purposes, most models of renewable resources management (see [5]). Indeed, our main purpose in this paper is to investigate how these models have to be modified to take into account intergenerational equity, and what the consequences are in terms of policy. So our first task is to develop the Sumaila-Walters approach in a continuous framework.

Consider a *public good*, available in quantity $x \geq 0$. Society is confronted with a scenario $x(t)$, where t ranges from 0 (today) to infinity (we are interested in the long term). Its members have to agree on a present value for this scenario.

We make the simplifying assumption that all individuals are identical: they all, born or unborn, have the same utility function $u(x)$. They also have the same rate of time preference δ. In other words, an individual born at time t attributes to the scenario $x(t)$, or rather to the portion of that scenario which he/she experiences, the utility:

$$\int_t^\infty e^{-\delta(s-t)} u(x(s))\, ds = \int_0^\infty e^{-\delta s} u(x(t+s))\, ds \tag{1}$$

We will also assume that the birth rate is α and the death rate $\omega < \alpha$, so that the growth rate is $n = \alpha - \omega > 0$. In other words, if the population at time 0 is N, the population at time t is Ne^{nt}; in the interval between t and $t+dt$, there will be $\alpha N e^{nt} dt$ births and $\omega N e^{nt} dt$ deaths. More precisely, of the generation born at time s, only a proportion $e^{-\omega(t-s)}$ is still alive at time $t > s$; so the total population at time t consists of $Ne^{(\alpha-\omega)t}$ individuals, $Ne^{-\omega t}$ of whom were alive at time 0, and $N\alpha e^{ns} e^{-\omega(t-s)} ds = N\alpha e^{-\omega t} e^{\alpha s} ds$ were born between s and $s + ds$, where $0 \leq s \leq t$.

We have to aggregate the utilities of all the generations: the present one, consisting of N individuals alive at time 0, and the future ones, who enter the fray at the rate α. As usual in the theory of social choice, we will use a *linear aggregator*, ascribing the coefficient $e^{-\sigma s}$ to the s-generation: in other words, each individual born between s and $s + ds$ carries the Pareto weight $e^{-\sigma s}$. At time t, the quantity of the public good is $x(t)$, and it affects all individuals who are still alive at that time, $Ne^{-\omega t}$ of whom were alive at time 0, and $N\alpha e^{ns} e^{-\omega(t-s)} ds = N\alpha e^{\alpha s} e^{-\omega t} ds$ of whom were born between s and $s + ds$. Each of them discounts the future at the constant rate δ, so that the aggregate utility evaluated at time 0 is:

$$\left(Ne^{-\omega t} e^{-\delta t} + \int_0^t e^{-\sigma s} N\alpha e^{-\omega t} e^{\alpha s} e^{-\delta(t-s)} ds \right) u(x(t)) = NR(t)\, u(x(t))$$

with:

$$R(t) = \lambda e^{-(\omega+\delta)t} + (1-\lambda) e^{-(\sigma-n)t}, \text{ with } \lambda = 1 - \frac{\alpha}{\alpha+\delta-\sigma} \text{ and } n = \alpha - \omega \tag{2}$$

The intertemporal welfare function $W(x)$ then is:

$$W(x) = \int_0^\infty R(t)\, u(x(t))\, dt \tag{3}$$

From now on we assume that

$$\sigma > n$$

The discount factor given by formula (2) corresponds to a non-constant discount rate

$r(t)$. We have:

$$r(t) := -\frac{R'(t)}{R(t)} = \frac{(1-\lambda)(\sigma-n) + \lambda(\omega+\delta)\, e^{(\sigma-\alpha-\delta)t}}{(1-\lambda) + \lambda e^{(\sigma-\alpha-\delta)t}} \tag{4}$$

$$r(t) \longrightarrow \delta - n \quad \text{when} \quad t \longrightarrow 0 \tag{5}$$

$$r(t) \longmapsto \min(\sigma - n, \omega + \delta) \quad \text{when } t \longrightarrow \infty \tag{6}$$

Note that the short-term rate is independent of σ: it is equal to $\delta - n$, where δ is the pure rate of time preference, and $n = \alpha - \omega$ is the growth rate of the population. Note also that it is *negative* if $\delta < n$.

When $\sigma = \alpha + \delta$, so that $\omega + \delta = \sigma - n$, formula (2) breaks down. It has to be replaced by its limit when $\sigma \longrightarrow n$, namely:

$$R(t) = (1 + \alpha t)\, e^{-\rho t}, \quad \text{with } \rho = \omega + \delta = \sigma - n \tag{7}$$

leading to a non-constant discount rate:

$$r(t) = \rho - \frac{\alpha}{1 + \alpha t} \tag{8}$$

$$r(t) \longrightarrow \rho - \alpha = \delta - n \quad \text{when } t \longrightarrow 0 \tag{9}$$

$$r(t) \longrightarrow \rho \text{ when } t \longrightarrow \infty \tag{10}$$

We will refer to (2) as *a quasi-exponential discount of type I*, and to (7) as *a quasi-exponential discounts of type II*. Note that in the first case, $r(\infty) < r(0)$ when $\sigma < \delta$ and $r(\infty) > r(0)$ when $\sigma > \delta$. In the second case, $r(\infty) > r(0)$.

To sum up, if we attribute non-zero Pareto weights to future generations, then the present value of a stream $x(t)$ of goods, whether public or private, should be given by formula (3):

$$W(x) = \int_0^\infty R(t)\, u(x(t))\, dt$$

where $R(t)$ is a quasi-exponential discount factor, given by formula (2) or (7), both of which lead to a non-constant discount rate $r(t) = -R'(t)/R(t)$. It is well known that in that case, the decision-makers face a problem of *time inconsistency*: see [8] for a survey. To see why, consider two streams $x_1(s)$ and $x_2(s)$, starting at time $T > 0$, and suppose that at a time 0 the first one is preferred:

$$\int_T^\infty R(s)\, u(x_1(s))\, ds \geq \int_T^\infty R(s)\, u(x_2(s))\, ds \tag{11}$$

Now take a subsequent time t, with $0 < T < t$, and compare the present values at time t. Is it still the case that $x_1(s)$ is preferred to $x_2(s)$? In the exponential case, $R(s) = \exp(-rs)$, we have:

$$\int_T^{\infty} e^{-r(s-t)} u(x(s))\, ds = e^{rt} \int_T^{\infty} e^{-rs} u(x(s))\, ds$$

so that both sides of equation (11) are multiplied by a positive constant, and the inequality persists. However, if $R(t)$ is not an exponential (in particular, if it is quasi-exponential), the inequality may well be reversed. In other words, if x_1 seemed better than x_2 at time $t = 0$, it may well be that x_2 will seem better than x_1 at a later time $t > 0$.

The consequences for policy-making are considerable. Suppose society is seeking an optimal policy, that is, a feasible stream $\bar{x}$ that will maximize the intertemporal welfare, that is, the present value $W(x)$. Suppose such a policy is found and acted upon at time $t = 0$. As soon as some time has elapsed, the decision-maker (the government, say) will find that $\bar{x}$ no longer maximizes present value on the remaining interval; in other words, at time $t > 0$, the optimal policy is some $\tilde{x} \neq \bar{x}$. It is to be expected that the decision-maker at time t will implement $\tilde{x}$ and not $\bar{x}$ if she is free to do so. As a result, there is no way for the decision-maker at time 0 to achieve what is, from her point of view, the first-best solution of the problem, and she must turn to a second-best policy. The best she can do is to guess what her successors are planning to do, and to lay down her own plan accordingly. In other words, we will be looking for a subgame-perfect equilibrium of a certain game in continuous time. This has been done by Ekeland and Lazrak (see [6] and [7]) in the framework of pure competition between decision-makers, and will be explained in the next section.

The plan of the paper is as follows. In Section 2, we define equilibrium strategies in the framework of the classical Schaefer model, and we prove that the classical results on optimal fisheries management still hold with non-constant discount rates, provided the pure rate of time preference δ is replaced by $\delta - n$, where n is the growth rate of the human population, and the notion of "optimal" strategy (which then useless) is replaced by the notion of "equilibrium" strategy (which is given in Definition 1). These results are stated in Theorems 2 and 3, and are proved in the Appendix.

2 Fisheries management

2.1 The Schaefer model for fisheries

A simple model of optimal fisheries management, probably originating with Schaefer [16], consists of seeking the optimal catching policy $h(t)$ as the solution of the problem:

$$\begin{aligned} &\max_h \int_0^\infty e^{-\delta t}(p-c(x(t)))h(t)dt \\ &\frac{dx}{dt} = f(x) - h(t), \quad 0 \le h \le h_{\max} \\ &x(0) = x_0 \end{aligned} \tag{Opt}$$

where $\delta > 0$ is the individual rate of time preference, p is the price of fish sold on shore, $c(x) > 0$ is the cost of bringing one fish to shore when the total stock (consisting of only one species) is x, and $f(x)$ models the natural growth of the stock. The cost c is a decreasing function of x. The fishing rate (control variable) is $h(t)$, which is bounded above by $h_{\max}$. We assume $f(0) = 0$, so that if $x(T) = 0$ for some T, meaning that the fish stock has been driven to extinction, there can be no recovery: from then on, we obviously stop the fishing rate: $h(t) = 0$ for $t > T$. Mathematicaly speaking, this is an optimal control problem, where $h(t)$ is the control and $x(t)$ is the state at time t. Schaefer himself took the specification:

$$f(x) = rx\left(1 - \frac{x}{K}\right), \quad c(x) = \frac{c}{qx} \tag{12}$$

but many other choices are possible. We refer to the book by Clark [5] for a thorough discussion of this problem and its variants. It is shown that the optimal catching policy $\tilde{h}(t)$ consists of bringing the stock as quickly as possible to a certain size $\tilde{x}$ and maintaining it from then on. Specifically, consider the equation:

$$f'(x) - \frac{c'(x)}{p - c(x)} f(x) = \delta \tag{13}$$

If it has a positive solution $\tilde{x}$, the optimal catching policy is given by:

$$\tilde{h}(t) = \begin{cases} 0 & \text{if } 0 \le x(t) < \tilde{x} \\ f(\tilde{x}) & \text{if } x(t) = \tilde{x} \\ h_{\max} & \text{if } x(t) > \tilde{x} \end{cases} \tag{14}$$

If there are no positive solutions, then the optimal catching policy consists of taking $\tilde{x} = 0$, that is of bringing all the fish stock to shore as quickly as possible. Generally speaking, using a high rate of time preference will result in driving the stock to extinction: see for instance Clark [3].

In the sequel, we will refer to strategies of type (14), optimal or not, as *threshold strategies*.

2.2 When optimization fails: equilibrium strategies

In line with the preceding section, we will now replace the exponential discount $e^{-\delta t}$ by a quasi-exponential discount factor $R(t)$. Let us assume that $R(t)$ is of type 1:

$$R(t) = \lambda e^{-(\omega+\delta)t} + (1-\lambda)\, e^{-(\sigma-n)t}, \text{ with } \lambda = 1 - \frac{\alpha}{\sigma-\alpha-\delta} \text{ and } \sigma > n \tag{15}$$

where $\sigma > 0$ expresses the concern about intergenerational equity: the lower σ is, the more weight is given to future generations. In the limiting case $\sigma = 0$, all generations are treated equally. Problem (Opt) now becomes:

$$\begin{gathered} \max_h \int_0^\infty R(s)\,(p - c(x(s)))h(s)ds \\ \frac{dx}{ds} = f(x) - h(s), \quad 0 \leq h \leq h_{\max} \\ x(0) = x_0 \end{gathered} \tag{Eq}$$

As discussed in the preceding section, the optimization problem can be still solved mathematically. The optimal solution will no longer be a threshold strategy, and it will be time-inconsistent: if one starts with the solution $\tilde{h}_0$ to problem (Eq), implement it to time $t > 0$, thereby reaching a point $\tilde{x}(t)$, and then computes anew the optimal fishing rate on $s \geq t$ starting from $x_t = \tilde{x}(t)$, one will find some $\bar{h}(s) \neq \tilde{h}(s)$. If there is no commitment mechanism, the first-best solution for the decision-maker at time 0 cannot be implemented and he/she will have to find a second-best solution.

This will be done by considering the situation no longer as an optimization problem, but as a non-cooperative game between successive decision-makers: an equilibrium strategy will be a Nash equilibrium of this leader-follower game. To be able to compute them, we will assume *perfect competition betwen decision-makers*: at every instant t, a new one assumes power, and will hold it for a vanishingly small amount of time ε. This way of using the infinitesimal properties of the continuum to model perfect competition goes back to Aumann in the case of markets, and has been introduced by Ekeland and Lazrak [6] in the context of time-inconstent optimization problems in continuous times (see [6] and [7])

To formalize the idea in the particular context of the Schaefer model, we need a few notations:

- Given some $x_\infty > 0$, we say that $h(t)$ is a *threshold strategy converging* to x_∞ (shortened to x_∞-strategy) if:

$$h(t) = \begin{cases} 0 & \text{if } 0 \le x(t) < x_\infty \\ f(x_\infty) & \text{if } x(t) = x_\infty \\ h_{\max} & \text{if } x(t) > x_\infty \end{cases} \tag{16}$$

- Denote by $\xi(t, h, x)$ the stock at time t,when the catching policy is $h(t)$ and the initial stock is x. The *present value* V of the threshold strategy h, given that the initial stock is x, is given by:

$$V(h, x) = \int_0^\infty R(s)\,(p - c(\xi(t, h, x)))h(t)dt$$

- Given a fishing rate $h(s)$, a time t, some $a \in R$ and some $\varepsilon > 0$, we define the (ε, t, a)-*perturbation* of h by:

$$h^\varepsilon(s) = \begin{cases} h(t) & \text{if } s \notin [t, t+\varepsilon] \\ a & \text{if } t \le s \le t+\varepsilon \end{cases}$$

Definition 1 *A threshold strategy* $h(t)$ *is an* equilibrium *if for every* (x, t, a)*, with* $0 \le a \le h_{\max}$*, we have*

$$\lim_{\varepsilon \longrightarrow 0} \frac{1}{\varepsilon}\left[V(h^\varepsilon, x)) - V(h, x)\right] \le 0 \tag{17}$$

where h^ε *is the* (ε, t, a)*-perturbation of* h*. It is a* local equilibrium *if (17) holds for all* x *in some neighbourhood of* x_∞*, all* $t \ge 0$ *and all* a *with* $0 \le a \le h_{\max}$

The interpretation is straightforward. Suppose an equilibrium strategy $h(x)$ is public knowledge, and has been implemented until time $t > 0$, leading to a situation $x_t = \xi(t, h, x)$. The decision-maker at time t reexamines the situation: his first-best solution cannot be implemented, as we have just seen, so he looks for a second-best. He will be in power between t and $t + \varepsilon$, and during that small interval of time he can exert any fishing rate h_0. After that, decisions will be made by others, and his best guess is that they will revert to the original strategy $h(x)$. This means that, if he chooses a, instead of applying $h(x_t)$, he will be changing the strategy h for the (ε, t, a)-*perturbation* h^ε of h, and the present value to him of doing that is $V(h^\varepsilon, x_t)$. On the other hand, the present value to him of applying $h(x_t)$ like everybody else is $V(h, x_t)$. We want $V(h^\varepsilon, x_t) \le V(h, x_t)$, so there is no incentive to him for defecting from the agreed strategy $h(x)$. But the difference $V(h^\varepsilon, x_t) - V(h, x_t)$ is clearly first-order in ε, hence the condition (17).

In other words, this is a Nash equilibrium: there is no incentive for unilateral deviations; note that the equilibrium strategy itself is Markovian (it depends only on the current state x), but it is robust against non-Markovian deviations.

If an equilibrium strategy converging to x_∞ is *local*, then (17) holds only in some interval $]x_\infty - a, \quad x_\infty - b[$. This means that, for any point in that interval, the strategy h is proof against unilateral deviations. So, if the starting point $x(0)$ lies in that interval, playing that strategy h will keep all the following $x(t)$, $t > 0$, in that interval (because it converges to x_∞), so none of the future decision-makers will have an incentive to deviate.

Theorem 2 *Suppose a threshold strategy h converging to x_∞ is an equilibrium strategy. Then the equilibrium stock x_∞ solves the equation:*

$$f'(x_\infty) - \frac{c'(x_\infty)}{p - c(x_\infty)} f(x_\infty) = \delta - n \tag{18}$$

Note that the right-hand side is the short (spot) discount rate for quasi-exponential discounts, as evidenced in formula (5). The left-hand side is the same as in formula (13), but on the right-hand side the pure rate of time preference of individuals is decreased by n, the growth rate of the population. There are two special cases:

$$R(t) = \lambda e^{-(\omega+\delta)t} + (1-\lambda) e^{-(\sigma-n)t}, \text{ with } \lambda = 1 - \frac{\alpha}{\sigma - \alpha - \delta} \text{ and } \sigma > n$$

The case when $\alpha = \omega = 0$. This means that the present generation is infinitely-lived and does not reproduce. In that case, there is obviously no concern about intergenerational equity, because there are no future generations to worry about. Indeed, plugging these values into (15), we find $\lambda = 1$ and $R(t) = e^{-\delta t}$, so we are back in the case of constant discount rate. This is precisely the framework of optimal management, as in [5], and recover precisely formula (13).

The case when $\alpha = \omega \neq 0$. This means that the population remains constant ($n = 0$), and is renewed at the rate $\alpha = \omega$. Plugging these values into (15), we find that $\lambda = \frac{\sigma-\delta}{\sigma-\alpha-\delta}$ and $R(t) = \lambda e^{-(\alpha+\delta)t} + (1-\lambda) e^{-\sigma t}$. The right-hand side then is still equal to δ, so that the formula (13) still holds. Note however that the corresponding equilibrium strategy cannot be optimal. Indeed, the discount rate is not constant, so the optimal strategy (from the point of view of time 0) must be time-dependent, and would probably not even be a threshold strategy.

Condition (18) is a simple generalisation of condition (13); it just states that, to take into account intergenerational equity, the rate of time preference δ on the right-hand side should be corrected by n, the growth rate of the population. It is a necessary condition, which singles out one (or several) candidate for the equilibrium stock. We will now prove that, under very natural assumptions, it is also sufficient.

Theorem 3 *Let $x_\infty > 0$ be a stock level satisfying (18), with $0 < f(x_\infty) < h_{\max}$ and $c(x_\infty) < p$. Suppose $f'(x_\infty) < 0$. Suppose moreover that, in some neighbourhood of x_∞, the function $c(x)$ is decreasing and the function $f(x)(p - c(x))$ is concave. Then there is a local equilibrium strategy h, converging to x_∞.*

Let us now take the Schaefer specification, $f(x) = rx\left(1 - \frac{x}{K}\right)$ and $c(x) = \frac{c}{qx}$. Condition (18) becomes:

$$2\frac{r}{K}\kappa x^2 - \left[\kappa(r - \delta + n) + \frac{r}{K}\right] x = \delta - n + r \quad \text{with} \quad \frac{pq}{c} = \kappa \tag{19}$$

Corollary 4 *Let $0 < x_\infty < K$ be a stock level satisfying (19), with $rx_\infty\left(1 - \frac{x_\infty}{K}\right) < h_{\max}$ and $x_\infty < \kappa$. If $x_\infty > K/2$, then there is a local equilibrium strategy h, converging to x_∞.*

Proof. We have $f(x)(p - c(x)) = \left(1 - \frac{x}{K}\right)\left(prx - \frac{rc}{q}\right)$, which is clearly a concave function. We also have $f'(x) = r\left(1 - \frac{2}{K}x\right)$, which is negative if $x_\infty > K/2$. The result then follows from Theorem 2. ■

Corollary 5 *Let $0 < x_\infty < K$ be a stock level satisfying (19), with $rx_\infty\left(1 - \frac{x_\infty}{K}\right) < h_{\max}$ and $x_\infty < \kappa$. If $|\delta - n|$ is small enough, then there is a local equilibrium strategy h, converging to x_∞.*

Proof. If $\delta = n$, then equation (19) has two roots, $x_\infty = 0$ and $x_\infty = \frac{1}{2}\left(K + \frac{1}{\kappa}\right) > K/2$. By continuity, for $|\delta - n|$ small, there will still be a root x_∞ larger than $K/2$ and the result will follow from the preceding corollary. ■

The proofs of Theorems 2 and 3 are deferred to the Appendix

3 Equitable discounting

3.1 Implementing equitable discounting

How will the planner implement an equilibrium strategy ? An individual alive at time t cannot be expected to internalize intergenerational equity. They will use their rate of time preference δ to discount future benefits, so that they will aim for a stationary stock x'_∞ given by:

$$f'(x'_\infty) - \frac{c'(x'_\infty)}{p - c(x'_\infty)} f(x'_\infty) = \delta \tag{20}$$

instead of the equilibrium population x_∞ given by (18)

The government can implement x_∞ by lowering the price of fish, that is by taxing the catch. Let r be the unit tax, so that the price after tax is $p - r$. The tax has to be such that solving (20) with $p - r$ instead of p leads to x_∞. This gives:

$$f'(x_\infty) - \frac{c'(x_\infty)}{p - r - c(x_\infty)} f(x_\infty) = \delta$$

and hence, after some calculations taking (18) into account:

$$r = n \frac{p - c(x_\infty)}{\delta - f'(x_\infty)}$$

Note that, if $c'(x_\infty) < 0$, which is a natural condition, then the second term on the right-hand side of (18) is positive, so that $f'(x_\infty) < \delta - n$. It follows that $0 < r < p - c(x_\infty)$: the tax rate is positive and fishing is still profitable.

3.2 The effect of equitable discounting

Using the Schaefer specification, we show graphically how the optimal fish population, x_∞, depends on the discount rate, δ, and the human population growth rate, n, (in formula (18)). We create graphs for fast growing (e.g., herrings) and slow growing (e.g., orange roughy) fish species, respectively, using the traditional and our new discounting approaches, more precise, we haved calculated the equilibrium population for a high growth fish specie with respect the discount rate for 3 cases: the traditional approach, our approach when the human population growths slowly, and our approach when it grows quickly.

We present in Figure 1 a graph showing how the optimal fish population changes for slow growing fish depending on the discounting approach used and the rate of growth of the human population. Figure 2 presents a similar graph for fast growing fish.

Some key observations from the figures are:

- Our new approach results in higher optimal fish populations for all discount rates;
- The difference between the optimal fish population under traditional discounting and our new discounting approach is largest for slow-growing fish, especially when the human population growth is high;

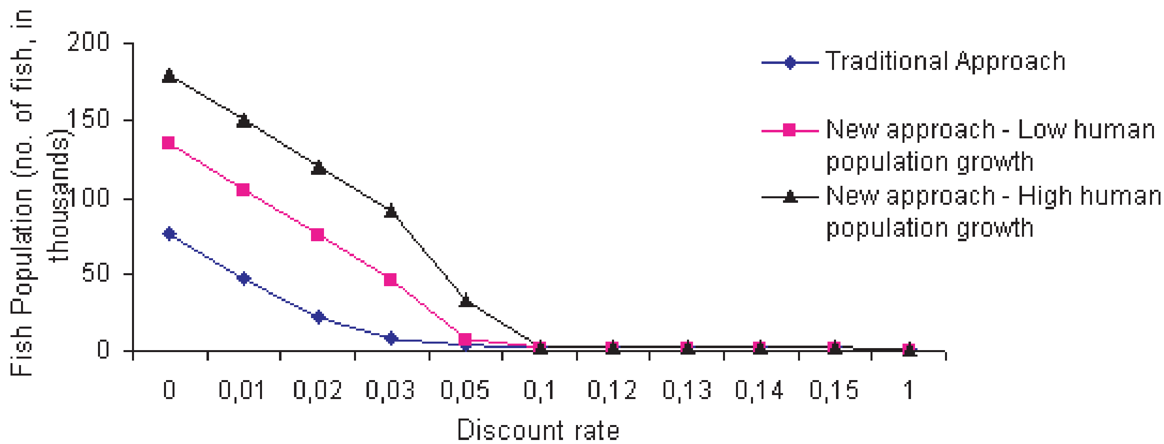

Figure 1: Optimal standing fish population for slow growing fish depending on the discounting approach used and the rate of growth of the human population.

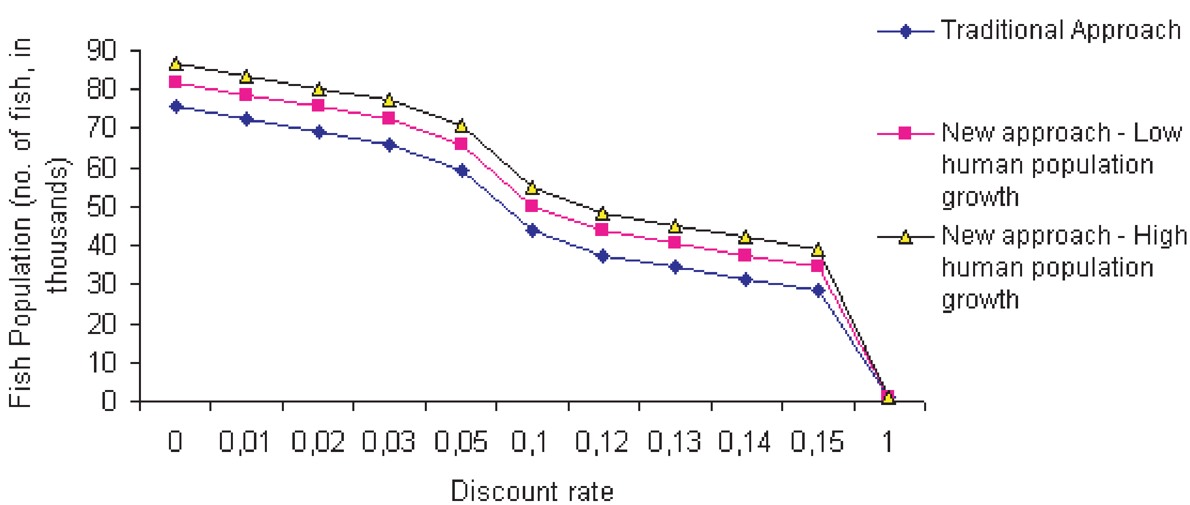

Figure 2: Optimal standing fish population for fast growing fish depending on the discounting approach used and the rate of growth of the human population.

- Because human population growth hardly exceeded 5% (globally) recently, the effect of our new approach declines as discount rates approach 5% in the case of slow growing fish and 15% in the case of fast growing fish.

4 Conclusion

We have shown that, in a very simple context where the human population grows at a constant rate n and all individuals have the same rate of time preference $\delta > 0$, the main conclusion of the Schaefer model still holds, with δ, the individual rate of time preference, replaced by $\delta - n$. This is a robust conclusion: indeed, it was reached by attaching a Pareto weight $e^{-\sigma t}$ to the generation born at time t, and it turns out to be independent of σ. It is also a reasonable one. When $n = 0$ for instance, so that the population is constant, each death being exactly balanced by one birth, so that whenever one individual disappears, an identical one arises in his/her place. It is then natural to consider this dynasty as a single individual living for ever, and having a rate of time preference equal to δ. Of course, one then forgets that transfers across time are really transfers across individuals. Our result shows that it is of no consequence for the equilibrium stock x_∞.

Applying our new approach to slow and fast growing fish species under different assumptions of human population growth rates, we demonstrate that our new approach means (i) higher optimal fish populations for all discount rates, for both slow and fast growing fish; and (ii) the difference between the optimal fish population under traditional discounting and our new discounting approach is largest for slow-growing fish, especially when the human population growth rate is high.

Our argument extends beyond the example of fisheries to the management of renewable resources in general. Take forestry for instance. The stock of trees is a public good, and if one considers the value of the harvest to be a proxy for the value of that public good, one is led to the same problem, where x now is the stock of trees instead of the stock of fish. More generally, when evaluating long-term projects, the consequences of which will be felt over several generations, we feel it is inappropriate to discount costs and benefits at the market rate ρ, which reflect the time preference of individuals. To take into account the impact of such projects on future generations, one should lower the interest rate, and the present paper suggests that it should be replaced by $\rho - n$, where n is the growth rate of the human population.

Of course, this has been proved only in a special case, and there are other effects to take into account. Some of them also tend to lower the long-term interest rate; in that category, we find the various types of uncertainty (on the growth rate [20], on the model [21]) and the heterogeneity of the individual rates of time preference accross the population [11]. Others, such as the growth rate of consumption, go in the other direction: if we believe that our descendants will be wealthier than we are, then the discount rate we use for intertemporal welfare should be increased. In real-life situations, where there is likely to be an increase in consumption goods and a decrease in environmental goods, the balance will be hard to strike. The scope of this paper has been restricted to the impact of intergenerational equity, and there remains much work to be done.

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A Appendix: Introduction

In the second part of this appendix, we prove Theorems 2 and 3. In the first part, we show how to associate with any threshold strategy h two functions $v(x)$ and $w(x)$, very similar to the value function in optimal control, and we study their differentiability properties. These functions will be helpful in characterizing equilibrium strategies and will be used to prove Theorems 2 and 3.

From now on, we will simplify the notations by writing the discount factor $R(t)$ as follows:

$$R(t) = \lambda e^{-\delta' t} + (1-\lambda) e^{-\sigma' t}$$

with:

$$\delta' := (\delta + \omega)$$
$$\sigma' := (\sigma - n)$$
$$\lambda := 1 - \frac{\alpha}{\alpha + \delta - \sigma}$$

B Threshold strategies

Let $h(t)$ be a threshold strategy (not necessarily an equilibrium strategy) converging to x_∞. Denoting, as above, by $\xi(t,h,x)$ the stock at time t, when the fishing rate is $h(t)$ and the initial stock is x, we introduce two functions which will play a crucial role:

$$v(x) := \int_0^\infty \lambda e^{-\delta' t}(p - c(\xi(t,h,x)))h(\xi(t,h,x))dt \tag{21}$$
$$w(x) := \int_0^\infty (1-\lambda)e^{-\sigma' t}(p - c(\xi(t,h,x)))h(\xi(t,h,x))dt \tag{22}$$

so that the present value associated with a fishing rate h and the starting stock x, is $V(h,x) = v(x)+w(x)$. It is clear from the definition of a threshold strategy that v and w are continously differentiable at every $x \neq x_\infty$. The case $x = x_\infty$ is important and will be handled directly.

We will now give explicit formulas for the derivatives $v'(x)$ and $w'(x)$.

B.1 The case $x < x_\infty$

We have $h(x) = 0$ and the fish stock is increasing. Let a small time $\tau > 0$ elapse, so that the stock reaches the level $x + \varepsilon < x_\infty$, with $\varepsilon = f(x)\tau$. We have, up to first order in ε:

$$\begin{aligned} v(x) &= \int_0^\infty \lambda e^{-\delta' t}[p - c(\xi(t,h,x))]h(\xi(t,h,x))dt \\ &= \int_0^\tau \lambda e^{-\delta' t}[p - c(\xi(t))]h(\xi(t))dt + \int_{\tau_+}^\infty \lambda e^{-\delta' t}[p - c(\xi(t))]h(\xi(t))dt \\ &= 0 + e^{-\delta'\tau} \int_0^\infty \lambda e^{-\delta' t}[p - c(\xi(t))]h(\xi(t))dt \\ &= e^{-\delta'\tau} v(x+\varepsilon) = \left(1 - \delta' \frac{\varepsilon}{f(x)}\right)(v(x) + \varepsilon v'(x)) \end{aligned}$$

So

$$v'(x) = \frac{\delta'}{f(x)} v(x) \tag{23}$$

and likewise:

$$w'(x) = \frac{\sigma'}{f(x)} w(x) \tag{24}$$

Adding up, we find that

$$(v'(x) + w'(x)) f(x) = \delta' v(x) + \sigma' w(x) \tag{25}$$

B.2 The case $x > x_\infty$

We have $h(x) = h_{\max}$ and the fish stock is decreasing. Let a small time $\tau > 0$ elapse, so that the stock reaches the level $x - \varepsilon > x_\infty$, with $\varepsilon = (h_{\max} - f(x))\tau$. We have, up to first order in ε:

$$\begin{aligned} v(x) &= \int_0^\infty \lambda e^{-\delta' t}[p - c(\xi(t,h,x))]h(\xi(t,h,x))dt \\ &= \int_0^\tau \lambda e^{-\delta' t}[p - c(\xi(t))]h_{\max} dt + \int_\tau^\infty \lambda e^{-\delta' t}[p - c(\xi(t))]h(\xi(t))dt \\ &= \lambda[p - c(x)]h_{\max}\tau + e^{-\delta'\tau} v(x - \varepsilon) \\ &= \lambda[p - c(x)]h_{\max}\tau + (1 - \delta'\tau)(v(x) - \varepsilon v'(x)) \\ &= v(x) + \tau\left[\lambda(p - c(x))h_{\max} - \delta' v(x) - (h_{\max} - f(x))v'(x)\right] \end{aligned}$$

This leads to:

$$v'(x) = \frac{\delta' v(x)}{f(x) - h_{\max}} - \lambda \frac{p - c(x)}{f(x) - h_{\max}} h_{\max} \tag{26}$$

$$w'(x) = \frac{\sigma' w(x)}{f(x) - h_{\max}} - (1 - \lambda)\frac{p - c(x)}{f(x) - h_{\max}} h_{\max} \tag{27}$$

Adding up, we find that

$$\left(v'\left(x\right)+w'\left(x\right)\right)\left(f\left(x\right)-h_{\max}\right)=\delta' v(x)+\sigma' w(x)-h_{\max}\left(p-c\left(x\right)\right) \tag{28}$$

B.3 The case $x = x_\infty$

Start from a smaller stock $x_\infty - \varepsilon$, with $\varepsilon > 0$ small, and apply the strategy h. This means that the fishing rate is $h\left(t\right) = 0$ until the level x_∞ is reached again. This will happen after a time $\tau = \varepsilon / f\left(x_\infty\right)$, and then the stock is stabilized at that level. This leads to:

$$\begin{aligned} v(x_\infty-\varepsilon) &= \int_0^\infty \lambda e^{-\delta' t}(p-c(\xi(t,h,x_\infty-\varepsilon)))h(\xi(t,h,x_\infty-\varepsilon)))dt \\ &= \int_\tau^\infty \lambda e^{-\delta' t}(p-c(x_\infty))f(x_\infty)dt = \frac{\lambda}{\delta'}e^{-\delta'\tau}(p-c(x_\infty))f(x_\infty) \end{aligned}$$

where we have taken into account that $h(\xi(t,h,x_\infty-\varepsilon))) = 0$ for $0 \le t \le \tau$. Hence the left derivative:

$$v'_-(x_\infty) = \lambda(p-c(x_\infty))$$

In the same way, we compute the right derivative. This time, we start with a larger stock $x_\infty + \varepsilon$, with $\varepsilon > 0$ small, and we apply the fishing level $h\left(t\right) = h_{\max}$ until the level x_∞ is reached again. This will happen after at time τ given by $\left(h_{\max} - f\left(x_\infty\right)\right)\tau = \varepsilon$. We have:

$$\begin{aligned} v(x_\infty+\varepsilon) &= \int_0^\tau \lambda e^{-\delta' t}(p-c(\xi(t,h,x_\infty+\varepsilon)))h_{\max}dt + \int_\tau^\infty \lambda e^{-\delta' t}(p-c(x_\infty))f(x_\infty)dt \\ &= \lambda(p-c(x_\infty))h_{\max}\tau + e^{-\delta'\tau}\frac{\lambda}{\delta'}(p-c(x_\infty))f(x_\infty) \\ &= v\left(x_\infty\right) + \left[\lambda(p-c(x_\infty))h_{\max} - \lambda(p-c(x_\infty))f(x_\infty)\right]\tau \\ &= v\left(x_\infty\right) + \lambda(p-c(x_\infty))\left(h_{\max} - f\left(x_\infty\right)\right)\tau \end{aligned}$$

and substituting the value for τ, we get $v'_+(x_\infty) = \lambda(p - c(x_\infty))$, which proves that the right and left derivatives are equal, so that v is derivable at x_∞, with $v'\left(x_\infty\right) = \lambda(p - c(x_\infty))$.

On the other hand, we also have:

$$v(x_\infty) = \int_0^\infty \lambda e^{-\delta' t}(p-c(x_\infty))h(x_\infty))dt = \frac{\lambda}{\delta'}h\left(x_\infty\right)\left(p-c\left(x_\infty\right)\right)$$

(fishing rate maintains the stock at the level x_∞), so that $\lambda\left(p - c\left(x_\infty\right)\right)f\left(x_\infty\right) = \delta' v\left(x_\infty\right)$. A similar argument holds for w. We summarize, bearing in mind that $h\left(x_\infty\right) = f\left(x_\infty\right)$ in equilibrium:

$$v'(x_\infty) = \lambda(p - c(x_\infty)) \tag{29}$$

$$w'(x_\infty) = (1-\lambda)(p - c(x_\infty)) \tag{30}$$

$$v(x_\infty) = \frac{\lambda}{\delta'} f(x_\infty)(p - c(x_\infty)) \tag{31}$$

$$w(x_\infty) = \frac{1-\lambda}{\sigma'} f(x_\infty)(p - c(x_\infty)) \tag{32}$$

Note that:

$$v'(x_\infty) = \frac{\delta'}{f(x_\infty)} v(x_\infty) = \frac{\delta' v(x_\infty)}{f(x_\infty) - h_{\max}} - \lambda \frac{p - c(x_\infty)}{f(x_\infty) - h_{\max}} h_{\max}$$
$$w'(x_\infty) = \frac{\sigma'}{f(x_\infty)} w(x_\infty) = \frac{\sigma' w(x_\infty)}{f(x_\infty) - h_{\max}} - (1-\lambda) \frac{p - c(x_\infty)}{f(x_\infty) - h_{\max}} h_{\max}$$

so that (23), (24), (26) and (27) all hold at $x = x_\infty$. As a consequence, so do (25) and (28)

C Equilibrium strategies

C.1 Characterization

We now consider the (ε, t, a)-*perturbation* of h. Without loss of generality, we can assume that $t = 0$ (that is, we reset our watches if necessary), so that:

$$h^\varepsilon(s) = \begin{cases} h(x(t)) & \varepsilon < t \\ a & 0 \le t \le \varepsilon \end{cases} \tag{33}$$

Let us write v_ε and w_ε instead of v_{h^ε} and w_{h^ε}, so that $v_0 = v$ and $w_0 = w$. Keeping only first-order terms in ε, we have, at any point $x \neq x_\infty$:

$$\begin{aligned}
v_\varepsilon(x) &= \int_0^\infty \lambda e^{-\delta' t}[p - c(\xi(t, h^\varepsilon, x))] h^\varepsilon(\xi(t, h^\varepsilon, x)) dt \\
&= \int_0^\varepsilon \lambda e^{-\delta' t}[p - c(\xi)] h^\varepsilon(\xi) dt + \int_\varepsilon^\infty \lambda e^{-\delta' t}[p - c(\xi)] h^\varepsilon(\xi) dt \\
&= \lambda[p - c(x)] a\varepsilon + \int_0^\infty \lambda e^{-\delta'(t+\varepsilon)}[p - c(\xi(t+\varepsilon))] h(\xi(t+\varepsilon)) dt \\
&= \lambda[p - c(x)] a\varepsilon + e^{-\delta'\varepsilon} \int_0^\infty \lambda e^{-\delta' t}[p - c(\xi(t+\varepsilon))] h(\xi(t+\varepsilon)) dt \\
&= \lambda[p - c(x)] a\varepsilon + (1 - \delta'\varepsilon) v_h(\xi(t+\varepsilon)) \\
&= v_h(x) + \varepsilon [v_h'(x)(f(x) - a) - \delta' v_h(c) + \lambda(p - c(x)) a]
\end{aligned}$$

The term $(f(x) - a)$ comes from the fact that, if the fishing rate is a exerted during a period ε when the stock is x, then the new stock at the end of the period will be $x + (f(x) - a)\varepsilon$, up to first order. Similarly, we get:

$$w_\varepsilon(x) = w(x) + \varepsilon\left[w'(x)(f(x) - a) - \sigma' w(c) + (1-\lambda)(p - c(x))a\right]$$

We then introduce the Hamiltonian $H(x,a)$:

$$H(x,a) := (v'(x) + w'(x))(f(x) - a) - \delta' v(x) - \sigma' w(x) + (p - c(x))a \tag{34}$$

$$= a\left[(p - c(x)) - (v'(x) + w'(x))\right] + (v'(x) + w'(x)) f(x) - \delta' v(x) - \sigma' w(x) \tag{35}$$

Condition (17) then reduces to the following:

$$\max\{H(x,a) \mid 0 \leq a \leq h_{\max}\} \leq 0 \tag{36}$$

By definition, $h(x)$ is an equilibrium strategy if and only if it satisfies condition (36). It is reminescent of the classical Hamilton-Jacobi-Bellman equation in optimal control, so once again we emphasize that, in the present situation, with non-constant discounting, it will NOT give an optimal solution, but an equilibrium one.

In (36) we find ourselves maximizing a linear function of a, so the maximum must be attained at the boundary unless the slope is zero. There are two possible cases for $x \neq x_\infty$, according to the value of the maximand $h(x)$:

- if $h(x) = 0$, so that $x < x_\infty$, the slope must be negative or zero:

$$0 \geq (p - c(x)) - (v'(x) + w'(x)) \tag{37}$$

- if $h(x) = h_{\max}$, so that $x > x_\infty$, the slope must be positive or zero:

$$0 \leq (p - c(x)) - (v'(x) + w'(x)) \tag{38}$$

C.2 Necessary condition: Theorem 2 .

We have proved that the function v and w are continuously differentiable everywhere. Conditions (37) and (38) mean that function $\varphi(x) := v'(x) + w'(x) - p + c(x)$ goes from ≥ 0 to ≤ 0 when x increases through x_∞. It is continous, and hence much vanish at x_∞:

$$v'(x_\infty) + w'(x_\infty) = p - c(x_\infty)$$

We know that φ is differentiable all $x \neq x_\infty$, but we cannot assume that it is differentiable at x_∞. So we cannot claim that $\varphi'(x_\infty) \leq 0$. However, there is a sequence $x_n \longrightarrow x_\infty$ from the left ($x_n < x_\infty$) and a sequence $y_n \longrightarrow x_\infty$ from the right ($y_n > x_\infty$) such that $\varphi'(x_n) \leq 0$ and $\varphi'(y_n) \leq 0$:

$$v''(x_n) + w''(x_n) \leq -c'(x_n) \text{ and } v''(y_n) + w''(y_n) \leq -c'(y_n)$$

Let us work on the first equation. Differentiating (25) we have:

$$(v''(x_n) + w''(x_n))f(x_n) + (v'(x_n) + w'(x_n))f'(x_n) = \delta' v'(x_n) + \sigma' w'(x_n)$$

Combining with the preceding inequation, we get:

$$\frac{1}{f(x_n)}\left[\delta' v'(x_n) + \sigma' w'(x_n) - (v'(x_n) + w'(x_n))f'(x_n)\right] \leq -c'(x_n)$$

and taking the limit as $n \longrightarrow \infty$, we get:

$$\frac{1}{f(x_\infty)}\left[\delta' v'(x_\infty) + \sigma' w'(x_\infty) - (v'(x_\infty) + w'(x_\infty))f'(x_\infty)\right] \leq -c'(x_\infty) \tag{39}$$

Now let us work on the y_n. Differentiating (28) we have:

$$(v''(y_n)+w''(y_n))(f(x) - h_{\max}) + (v'(y_n)+w'(y_n))f'(y_n) = \delta' v'(y_n) + \sigma' w'(y_n) + h_{\max} c'(y_n)$$

Combining with the inequality $\varphi'(y_n) \leq 0$, and taking the limit as $n \longrightarrow \infty$, we get:

$$\frac{1}{f(x) - h_{\max}}\left[\delta' v'(x_\infty) + \sigma' w'(x_\infty) + h_{\max} c'(x_\infty) - (v'(x_\infty) + w'(x_\infty))f'(x_\infty)\right] \leq -c'(x_\infty) \tag{40}$$

$$\delta' v'(x_\infty) + \sigma' w'(x_\infty) - (v'(x_\infty) + w'(x_\infty))f'(x_\infty) \geq -f(x_\infty)\, c'(x_\infty)$$

Combining (39) and (40), we find:

$$\delta' v'(x_\infty) + \sigma' w'(x_\infty) - (v'(x_\infty) + w'(x_\infty))f'(x_\infty) = -f(x_\infty)\, c'(x_\infty)$$

Plugging in the values for $v'(x_\infty)$ and $w'(x_\infty)$ from (29) and (30), we find:

$$(p - c(x_\infty))\,(\lambda\delta' + (1-\lambda)\,\sigma' - f'(x_\infty)) = -f(x_\infty)\, c'(x_\infty) \tag{41}$$

and substituting the value for λ, we get formula (18)

C.3 General growth and cost: Theorem 3

As in section 3.3, we define v and w by (21) and (22). Differentiate equations (23) and (24) from the left at x_∞:

$$v''_-(x_\infty) = v'(x_\infty)\frac{\delta' - f'(x_\infty)}{f(x_\infty)} = \frac{\lambda(p - c(x_\infty))(\delta' - f'(x_\infty))}{f(x_\infty)}$$
$$w''_-(x_\infty) = w'(x_\infty)\frac{\sigma' - f'(x_\infty)}{f(x_\infty)} = \frac{(1-\lambda)(p - c(x_\infty))(\sigma' - f'(x_\infty))}{f(x_\infty)}$$

Set $I(x) = f(x)(p - c(x))$ and $\psi(x) = v(x)\delta' + w(x)\sigma' - I(x)$. Note that, by (29), (30), (31) and (32), we have:

$$\psi(x_\infty) = 0 = \psi'(x_\infty)$$

Now consider the (left) second derivative $\psi''_-(x_\infty)$.After some computations, we find:

$$\psi''_-(x_\infty) = \left(\delta'\frac{\lambda(p - c(x_\infty))(\delta' - f'(x_\infty))}{f(x_\infty)} + \sigma'\frac{(1-\lambda)(p - c(x_\infty))(\sigma' - f'(x_\infty))}{f(x_\infty)} - I''(x_\infty)\right)$$
$$= (\delta' A + \sigma' B - I''(x_\infty))$$

with obvious notations. Note that $A + B = -c'\,(c_\infty)$ by (41). Since $\sigma' > 0$ and $f'(x_\infty) < 0$ by hypothesis, follows that $AB \geq 0$; moreover $A + B \geq 0$ because $c\,(x)$ has been assumed to be decreasing. It follows that A and B are positive, and hence:

$$\psi''_-(x_\infty) \geq -\min(\delta', \sigma')c'(x_\infty) - I''(x_\infty) > 0$$

So there exist some $a < x_\infty$ such that $\psi(x) > 0$ for all x in the open interval $]a,\ x_\infty[$.

We redo the preceeding analysis but for $x > x_\infty$. We find:

$$v''_+(x_\infty) = \frac{\lambda c'(x_\infty)h_{\max} + \lambda(p - c(x_\infty))(\delta' - f'(x_\infty))}{f(x_\infty) - h_{\max}}$$
$$w''_+(x_\infty) = \frac{(1-\lambda)c'(x_\infty)h_{\max} + (1-\lambda)(p - c(x_\infty))(\sigma' - f'(x_\infty))}{f(x_\infty) - h_{\max}}$$

and hence:

$$\psi''_+(x_\infty) = -(p - c(x_\infty))\frac{\lambda\delta'(\delta' - f'(x_\infty)) + (1-\lambda)\sigma'(\sigma' - f'(x_\infty)}{\bar{h} - f(x_\infty)}$$
$$- \left(\frac{(\lambda\delta' + (1-\lambda)\sigma')c'(x_\infty)\bar{h}}{\bar{h} - f(x_\infty)} + I''(x)\right)$$
$$\geq -\left(\min(\sigma', \delta')c'(x_\infty) + I''(x_\infty)\right) > 0$$

As above, there exists $b > x_\infty$ such that $\psi(x) > 0$ for all x such that $b > x > x_\infty$. So, for all x in the interval $]a,\ b[$ we have $v(x)\delta' + w(x)\sigma' \geq f(x)(p - c(x)$. Using (23) and (24), this yields:

$$v'(x) + w'(x) = \frac{v(x)\delta' + w(x)\sigma'}{f(x)} \geq p - c(x) \text{ for } a < x < x_\infty$$

while using (26) and (27), we get:

$$v'(x) + w'(x) = \frac{-[p - c(x)]h_{\max} + \delta' v(x) + \sigma' w(x)}{f(x) - h_{\max}} \leq p - c(x) \text{ for } x_\infty < x < b$$

But these two inequalities are precisely (37) and (38). So the threshold strategy converging to x_∞ is an equilibrium strategy, as announced.