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Economic Calculus and Sustainable Development

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Abstract

This article discusses the discount rate to be used in projects that aimed at improving the environment. The model is quite stylized and involves two different goods, one is the usual consumption good whose production increases exponentially, the other is an environmental good whose quality remains limited.

We define an ecological discount rate and we exhibit different relations that link this discount rate with the usual interest rate and the growth rate. We also discuss, mainly in the long run, the sensibility of this ecological discount rate to the parameters.

Introduction

1 Model and preliminary insights

1.1 Goods and Preferences

We are considering a world with two goods. Each of them has to be viewed as an aggregate. The first one is the standard aggregate private consumption of growth models. The second one is called the environmental good. Its "quantity" provides an aggregate measure of "environmental quality" at a given time. It may be viewed as reflecting biodiversity, the quality of landscapes, nature and recreational spaces, the quality of climate. Later, however, for the sake of interpretation, we will view the index, as integrating, in a broader way, many non-markets dimensions of welfare.

We call x_t the quantity of private goods available at period t, and y_t the level of environmental quality at the same period. Generation t, that lives at period t only, has ordinal preferences, represented by a concave, homogenous of degree one utility function:

$$v(x_t, y_t) = \left[x_t^{\frac{\sigma-1}{\sigma}} + y_t^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

However, the measurement of cardinal utility involves an iso-elastic function.

$$V(x_t, y_t) = \frac{1}{1-\sigma'} v(x_t, y_t)^{1-\sigma'}$$

The above modelling calls for the following comments that concern respectively v and V.

- Concerning v, we stress the standard properties of a CES utility function, where σ is the elasticity of substitution, by referring to the consumer's choice if he were faced with a price both for the environmental good and the private good. (Here the price of the environmental good has to be viewed as an implicit price)
  - When the ratio (implicit) price of the environmental good over price of the private good decreases by one per cent, then the ratio quantity (here quality) of the environmental good over quantity of the private good increases by σ per cent. Equivalently, when the *ratio environmental quantity (here quality) over private good quantity decreases by one per cent the marginal willingness to pay for the environmental good increases by* $(1/\sigma)$ *per cent*

    It follows that if, as we often suppose in the following, environmental quality is constant and equals $\overline{y}$, and the private good consumption increases at the rate g, then the marginal willingness to pay for the environmental good increases at the rate (g/σ), which is greater, (resp. smaller) than g, if σ is smaller, (resp. greater) than one.
  - The reader has noted that both x_t and y_t appear with the same coefficient in the function v. However this is without loss of generality as soon as we keep control of the freedom in the measurement of y_t. Indeed, as we shall see later, one can define at each period a "green GDP", (the product of the implicit price of the environmental good by its quantity) and the standard GDP, (the product of the quantity of private goods by its price). The ratio of green GDP over the standard GDP is indeed, see below, $(\frac{y_t}{x_t})^{1-\frac{1}{\sigma}}$, and we may calibrate the model, i.e. choose the units of measurement of the environmental good by assessing the relative value of green GDP at the first period.
- Let us come to V. The marginal utility of a "util" of v, takes the form $v^{-\sigma'}$: when v increases by one per cent, marginal cardinal utility decreases by σ' per cent. This is the standard coefficient linked to intertemporal elasticity of substitution $(\frac{1}{\sigma'})$, relative risk aversion (σ') or intertemporal resistance to substitution.

1.2 Social welfare

Social welfare is evaluated as the sum of generational utilities. In line with the argument of Koopmans, we adopt the standard utilitarian criterion[1]:

[1] Notice that the elasticity σ' can be thought of as an ethical viewpoint of an external planner instead of the above interpretation.

$$\frac{1}{1-\sigma'}\sum_{t=0}^{+\infty} e^{-\delta t} v(x_t, y_t)^{1-\sigma'}$$

Two comments can be made:

- The coefficient δ is a pure rate of time preference. Within the normative viewpoint which we mainly stress here, the fact that this coefficient is positive has been criticized, for example by Ramsey who claims that this is "ethically indefensible and arises merely from the weakness of the imagination" or Harrod (1948) who views that as a "polite expression for rapacity and the conquest of reason by passion". To reconcile these feelings with Koopmans' argument[2], we say that "ethical considerations become preponderant" when δ tends to zero. We may view the number as the probability of survival of the planet[3].
- We may view the coefficient σ', as a purely descriptive one, reflecting intertemporal and risk behavior, or as a partly normative coefficient, reflecting the desirability of income redistribution across generations. In the following we will vary the interpretation according to the viewpoint.

2 Preliminaries: Investigation around a simple reference trajectory

2.1 The reference trajectory and first insights

In order to give some intuition on the discount rates question, we shall consider a reference trajectory of the economy where environmental quality is fixed at the level $\overline{y}$ and where the sequence of private goods consumption denoted x_t^* is also given (we often assume that the growth rate g of consumption is itself fixed)[4]. Note that the opposition between a finite level of environmental good and an increasing level of consumption good reflects an essential feature of the ecological question: sites, species are finite and up to now we only have a planet where modern optimism leads to believe that consumption goods may be multiplied without limit. Note that our formulation, at least at this stage, does not assume either "limits to growth" due to the finite ecological resources nor even deterioration of the ecological production due to growth. But it leaves open the degree of substitutability between standard consumption good and environment. We will argue later that this is one key uncertainty of the problem, although at this stage, we leave the elasticity of substitution σ as fixed.

[2] "Overtaking" would be another, different, way to proceed.

[3] This is more satisfactory when we model adequately the uncertainty of the problem, than within a deterministic framework where a higher δ may sometimes be a proxy for the inappropriate treatment of uncertainty.

[4] These sequences can be thought of as resulting from an optimization problem with exogenous progress. However, models à la Ramsey-Solow do not necessarily lead to a stationary equilibrium (for a review on these issues, see Guesnerie-Woodford (1992))

The question we examine is: What are the discount rates, standard interest rate for private goods, i.e. the return to private capital r_t, and the ecological rate for environmental goods implicit to the fixed trajectory?

At this stage, one may question the long run validity of the plausible short run hypothesis that marginal willingness or environmental amenities grows faster than private wealth, i.e. that $\sigma < 1$. At this stage, we will not decide about the value of σ, but will take for granted that it is temporally stable. We shall come back on this assumption[5].

2.2 Implicit discount rates

We shall first investigate the implicit discount factors at the margin of our reference trajectory, with fixed environmental quality and exponential growth. We sometimes refer to this approach as the reform viewpoint. In the next section, we shall then take the optimization viewpoint and show conditions under which our reference trajectory is a first best social optimum.

The reference trajectory * has consumption growing at the rate g_t^* (by definition $x_{t+1}^* = e^{g_t^*} x_t^*$) and the environmental quality equal to $\overline{y}$.

We want to compute the implicit discount rates that sustain this trajectory, that is the discount rates that make it locally optimal.

Definition 1. *The implicit discount rate for private good between periods t and $t+1$, is r_t^* such that*

$$e^{-r_t^*} = e^{-\delta} \frac{\partial_1 V(x_{t+1}, \overline{y})}{\partial_1 V(x_t, \overline{y})}$$

where $\partial_1 V(x, y) = \left[x^{\frac{\sigma-1}{\sigma}} + y^{\frac{\sigma-1}{\sigma}}\right]^{\left(\frac{1-\sigma\sigma'}{\sigma-1}\right)} x^{-\frac{1}{\sigma}}$.

As in finance, it is possible to define the discount rate between periods 0 and T with the sequence of short-term rates:

$$R^*(T) = \frac{1}{T} \sum_{t=0}^{T-1} r_t^*$$

Now, we can define by analogy our new notion.

Definition 2. *The ecological implicit discount rate between two consecutive periods is β_t^* defined by:*

$$e^{-\beta_t^*} = e^{-\delta} \frac{\partial_2 V(x_{t+1}, \overline{y})}{\partial_2 V(x_t, \overline{y})}$$

[5] It's arguably difficult to guess the right value for σ because it is not only the elasticity of substitution between the two goods but also a complete characterization of preferences.

The discount rate between periods 0 *and* T *is:*

$$B^*(T) = \frac{1}{T}\sum_{t=0}^{T-1}\beta_t^*$$

Proposition 1. *Along our reference trajectory,* $x_0^*, \dots, x_{t+1}^* = e^{g_t^*}x_t^*, \dots$ *the implicit private discount rate for the private good between periods* t *and* $t+1$ *can be equivalently written as,*

either:

$r_t^* = \delta + g_t^*\sigma' + \frac{1-\sigma\sigma'}{\sigma-1}\ln\left(\frac{1+\rho_t^*}{1+\rho_{t+1}^*}\right)$

or

$r_t^* = \delta + g_t^*/\sigma + \frac{1-\sigma\sigma'}{\sigma-1}\ln\left(\frac{1+\rho_t^{*\,-1}}{1+\rho_{t+1}^{*\,-1}}\right)$

where $\rho_t = \frac{y_t\partial_2 V}{x_t\partial_1 V} = \left(\frac{x_t}{y_t}\right)^{\frac{1}{\sigma}-1}$ *is the ratio of Green GDP over standard GDP, and* ρ_t^* *is its value along the trajectory.*

Proposition 2. *The ecological discount rate between periods* t *and* $t+1$ *is:*

$$\beta_t^* = r_t^* - g_t^*/\sigma$$

2.3 Long run discount rates: the reform viewpoint

Now we stress the long run behavior of the discount rates, under the assumption that the average growth rate of consumption converges: $\frac{1}{T}\sum_{t=0}^{T-1} g_t^* \to g^*$.

We focus our attention on the long run discount rate for private good, i.e. the limit of the discount rate between periods 0 and T, $R^*(T) = \frac{1}{T}\sum_{t=0}^{T-1} r_t^*$, when T becomes high. Similarly, the long run ecological discount rate is the limit, when T increases indefinitely of $B^*(T) = \frac{1}{T}\sum_{t=0}^{T-1}\beta_t^*$

At this stage, one should give some intuitions on the qualitative differences between the cases $\sigma < 1$ and $\sigma > 1$ that will appear strikingly in the results.

We have:

$$v(x_t, y_t) = x_t[1 + (\frac{y_t}{x_t})^{(\frac{\sigma-1}{\sigma})}]^{(\frac{\sigma}{\sigma-1})}$$

First, let us consider $\sigma > 1$[6]. Now, v grows as x_t whenever $\frac{y_t}{x_t}$ tends to zero and social marginal utility of consumption will decrease at σ' times the growth rate of v.

[6] For example, with $\sigma = 2$

$$v(x_t, y_t) = x_t[1 + (\frac{y_t}{x_t})^{(1/2)}]^2$$

On the contrary, in the case where $\sigma < 1$[7] it is useful to write:

$$v(x_t, y_t) = y_t[1 + (\frac{y_t}{x_t})^{(\frac{1-\sigma}{\sigma})}]^{(\frac{\sigma}{\sigma-1})}$$

In that case v does not grow any longer indefinitely with x_t, but tends to $\overline{y}$. Then, social marginal utility of consumption is approximately equal to $(\overline{y})^{-\sigma'}$ and tends to zero at a speed independent of σ'.

These differences are reflected in a particularly spectacular way in the behavior of long run discount rates.

Proposition 3. *At the margin of the reference situation, when T tends to $+\infty$,*

- When $\sigma > 1$,
$R^*(T) \to \delta + g^*\sigma'$
$B^*(T) \to \delta + g^*(\sigma' - 1/\sigma)$

- When $\sigma < 1$,
$R^*(T) \to \delta + g^*/\sigma$
$B^*(T) \to \delta$

This proposition is really central in our paper and need some comments:

- The cases $\sigma < 1$ and $\sigma > 1$ are really different in the sense that asymptotic results are discontinuous at $\sigma = 1$. The first case ($\sigma > 1$) can be thought of as a traditional case since the two goods are good substitutes. The second case ($\sigma < 1$) is characterized by a low substituability between the private good and the environmental good and corresponds to a case where environmental issues become preponderant in the long run.
- In the $\sigma < 1$ case the asymptotic ecological discount rate only depends on δ and not on the growth path. This is one of the most important properties in this part since it may be an incitation to take very low discount rate for the environmental good.

The $\sigma = 1$ case

For the sake of completeness, we have to consider the $\sigma = 1$ case. For this case the utility function is not defined but we can use a very common trick that is: $\lim_{\sigma\to 1} \frac{1}{2^{\frac{\sigma}{\sigma-1}}} v(x, y) = \sqrt{xy}$.

[7] For $\sigma = 1/2$,

$$v(x_t, y_t) = y_t[1 + (\frac{y_t}{x_t})]^{(-1)}$$

Consequently, we can use the same definitions and reasonings as those we used but with a Cobb-Douglas function. The results are the following:

Proposition 4. *If $\sigma = 1$, at the margin of the reference situation, when T tends to $+\infty$,*
$R^*(T) \to \delta + \frac{1}{2}g^*(\sigma' + 1)$
$B^*(T) \to \delta + \frac{1}{2}g^*(\sigma' - 1)$

The case of environmental good exhaustion

We can also generalize our result by making an other hypothesis on y_t^*. We can indeed consider an exhaustion of the environmental good at a positive rate g'. This means that the condition $y_t^* = \overline{y}$ is replaced by $y_t^* = \overline{y}e^{-g't}$.

With this setting, the above propositions are slightly modified but we can use the same methodology to get the asymptotic results on the two interest rates:

Proposition 5. *If g' is not too large, at the margin of the reference situation, when T tends to $+\infty$,*

- When $\sigma > 1$,
$R^*(T) \to \delta + g^*\sigma'$
$B^*(T) \to \delta + g^*(\sigma' - 1/\sigma) - \frac{g'}{\sigma}$

- When $\sigma < 1$,
$R^*(T) \to \delta + g^*/\sigma + g'\frac{1-\sigma\sigma'}{\sigma}$
$B^*(T) \to \delta - \sigma' g'$

Robustness

We said before that the parameter σ was hard to calibrate. One can therefore ask whether the result is still true in the case where σ is not constant but depends on t[8].

Proposition 6. *If $\lim_{t\to+\infty}\sigma_t = \sigma \neq 1$ then, at the margin of the reference situation, when T tends to $+\infty$,*

- When $\sigma > 1$,
$R^*(T) \to \delta + g^*\sigma'$
$B^*(T) \to \delta + g^*(\sigma' - 1/\sigma)$

- When $\sigma < 1$,
$R^*(T) \to \delta + g^*/\sigma$

[8] By this we mean that σ_t is a deterministic function (the random case being hard to tackle.)

$B^*(T) \to \delta$

The case where $\sigma_t \to 1$ is undetermined and depends, among other things, on the convergence speed.

3 Discount rates along an optimized growth path

The above results hold at the margin of any trajectory, whether it is non-optimal, or optimal either in a first best sense or in a second best sense. Optimal solutions of growth models, either first best or second best have the limit properties that we attributed to our reference solution and hence the results of Proposition 2 apply in a variety of worlds (naturally in a second best world the implicit discount rate for private good may differ from market prices). Hence in a sense, the above analysis exhausts, at least in a large class of contexts, the qualitative properties of long run ecological discount rates.

However, we will now turn to a fully defined optimization framework in order to say more on the whole trajectory of ecological discount rates. The model will also allow us to provide a better assessment of an a priori worrying discontinuity of our results as a function of σ.

We consider a model à la Keynes-Ramsey where the interest rate r is exogenous. This interest rate can come from an AK model or from a market that has no link with the model: a good example of this is given by r extracted from a *research arbitrage equation* as in Aghion Howitt.

3.1 The model and characterization of the first best social optimum

We consider a representative agent, living for ever, who maximizes the following intertemporal utility function:

$$\sum_{t=0}^{\infty} \exp(-\delta t) V(x_t, y_t)$$

where we assume, partly for convergence reasons, that $(1-\sigma')r < \delta < r$[9].

The representative agent, sometimes denoted below as the planner, has economic and environmental constraints:

Economic constraints: $a_{t+1} = \exp(r)a_t + w_t - x_t$, where a_t stands for the wealth at date t and w_t is an exogenous production flow,

Environmental constraints: The available quality for the environmental good is limited to $\overline{y}$ that is: $y_t \leq \overline{y}$.

[9] It's interesting to note that if $\sigma' > 1$, one can take negative values for the discount rate δ.

To solve the optimization problem we introduce the lagrangian associated to the problem and this gives:

$$\mathcal{L} = \sum_{t=0}^{\infty} \exp(-\delta t)[V(x_t, y_t) + \lambda_t(\exp(r)a_t + w_t - x_t - a_{t+1}) + \mu_t(\overline{y} - y_t)]$$

The first order conditions are the following:

$$\begin{cases} \partial_{x_t}\mathcal{L} = 0 \iff \partial_x V(x_t^*, y_t^*) = \lambda_t \\ \partial_{a_{t+1}}\mathcal{L} = 0 \iff \lambda_{t+1}\exp(r-\delta) = \lambda_t \\ \partial_{y_t}\mathcal{L} = 0 \iff \partial_y V(x_t^*, y_t^*) = \mu_t \end{cases}$$

From these first order conditions we can derive the asymptotic economic growth rate:

Proposition 7. *The asymptotic growth rate for the private good x_t^* depends on σ and is given by the following formulae:*

- If $\sigma < 1$ then $g_\infty^* = \sigma(r - \delta)$ and there is a real environmental issue.

- If $\sigma > 1$ then $g_\infty^* = \frac{r-\delta}{\sigma'}$ and this is the traditional result without any consideration of environmental goods.

- If $\sigma = 1$ then $g_\infty^* = \frac{2(r-\delta)}{1+\sigma'}$ and it is a specific case.

In this framework we really see the difference between the two cases $\sigma > 1$ and $\sigma < 1$. In the former case, the asymptotic growth is not affected by the presence of the environmental good whereas in the latter case, the growth rate is different from the usual one and decreases when σ gets lower.

Now, we can apply our finding to the ecological interest rates B^* (using Proposition 2 and Césàro's theorem). We have the following results:

Proposition 8. *The asymptotic ecological discount rate $B_\infty^* = \lim_{T\to+\infty} B^*(T)$ is given by the following formulae:*

- If $\sigma < 1$ then $B_\infty^* = \delta$.

- If $\sigma > 1$ then $B_\infty^* = (1 - \frac{1}{\sigma\sigma'})r + \frac{1}{\sigma\sigma'}\delta$.

- If $\sigma = 1$ then $B_\infty^* = \delta - \frac{1-\sigma'}{1+\sigma'}(r - \delta)$.

As before, we have to notice that the asymptotic ecological discount rate does not depend on economic considerations but depends only on preferences in the $\sigma < 1$ case.

3.2 More on the apparent discontinuity of the results with the elasticity of substitution σ

Both the general setting and the optimization setting we are now dealing with suffer from what can be thought of as a real caveat. The ecological interest rate seems indeed to be a discontinuous function of σ.

If we plot the asymptotic ecological interest rate as a function of σ we obtain the following result:

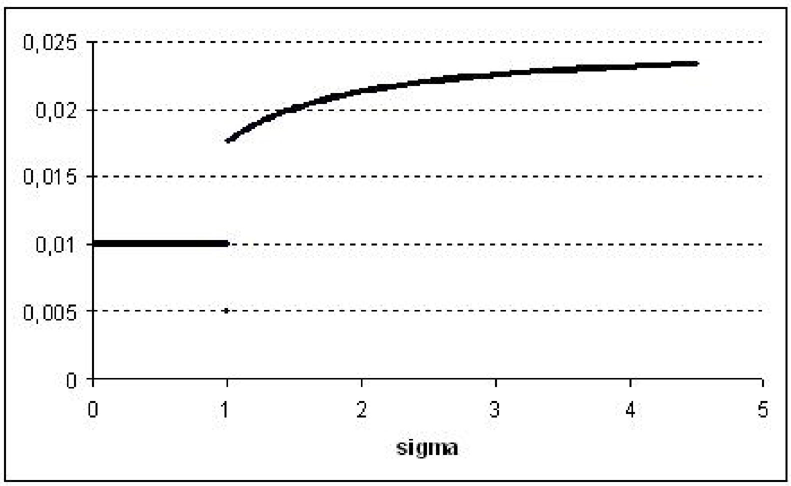

Figure 1: Dependence on σ of the variable B^*_∞ when $\sigma' = 2$

In what follows we show that this discontinuity is specific to the asymptotic case and that $B^*(T)$ is a continuous function of σ when T is fixed (and finite).

Proposition 9. $\forall T < \infty, \sigma \mapsto B^*(T;\sigma)$ *is continuous.*

This Proposition 9 is important in the sense that the discontinuity in Proposition 3 or Proposition 8 is proven to be specific to the asymptotic case. Therefore, this discontinuity is not really a problem in our model.

3.3 The dynamics of ecological discount rates

The Ramsey-Keynes framework is also interesting to deal with the ecological discount rates in finite horizon and to see the evolution of ecological discount rates with time. In other words, we are going to be interested in what can be called yield curves for ecological discount rates $B^*(T)$.

Since $B^*(T) = r - \frac{1}{\sigma}\frac{1}{T}\sum_{t=0}^{T-1} g_t^*$, we see that the dynamics of the ecological discount rate is linked to the dynamics of growth.

Therefore, we are going to focus on the way g_t^* converges toward its limit.

Proposition 10. *g_t^* converges monotonically toward its limit according to the following rules:*
- If $\sigma < 1$ and $\sigma\sigma' > 1$ then g_t^* is increasing.
- If $\sigma > 1$ and $\sigma\sigma' > 1$ then g_t^* is decreasing.
and less importantly:
- If $\sigma < 1$ and $\sigma\sigma' < 1$ then g_t^* is decreasing.
- If $\sigma > 1$ and $\sigma\sigma' < 1$ then g_t^* is increasing.
- If $\sigma = 1$ or $\sigma\sigma' = 1$ the optimal growth rate is constant.

The result of Property 10 is in fact the superposition of two effects.

First, there is a substitution effect between the two goods. If this substitution is very low, we are incited not to postpone consumption of the private good and therefore the growth is naturally decreasing (respectively increasing if σ is high).

Second, there is an intertemporal substitution effect. If the substitution between two periods is low (if σ' is high), the growth is supposed to be decreasing (respectively increasing if σ' is low).

Now using Proposition 2, we can deduce the shape of the yield curve for ecological discount rate:

Proposition 11. *The shape of the yield curve is the following:*
- If $\sigma < 1$ and $\sigma\sigma' > 1$ then $T \mapsto B^*(T)$ is decreasing.
- If $\sigma > 1$ and $\sigma\sigma' > 1$ then $T \mapsto B^*(T)$ is increasing.
and less importantly:
- If $\sigma < 1$ and $\sigma\sigma' < 1$ then $T \mapsto B^*(T)$ is increasing.
- If $\sigma > 1$ and $\sigma\sigma' < 1$ then $T \mapsto B^*(T)$ is decreasing.
- If $\sigma = 1$ or $\sigma\sigma' = 1$ then $T \mapsto B^*(T)$ is constant.

To illustrate our proposition, we drew yield curves using a simulation of the growth path[10]. Two examples are given below where the x-axis represents years and the y-axis the value of the ecological discount rate.

The first case corresponds to $\sigma < 1$ and $\sigma\sigma' > 1$ in which case the yield curve is decreasing and converges towards δ.

The second case corresponds to $\sigma > 1$ and $\sigma\sigma' > 1$ in which case the yield curve is increasing and converges towards $r - \frac{r-\delta}{\sigma\sigma'}$.

[10] We simply used a Newton-Raphson methodology to find the growth path

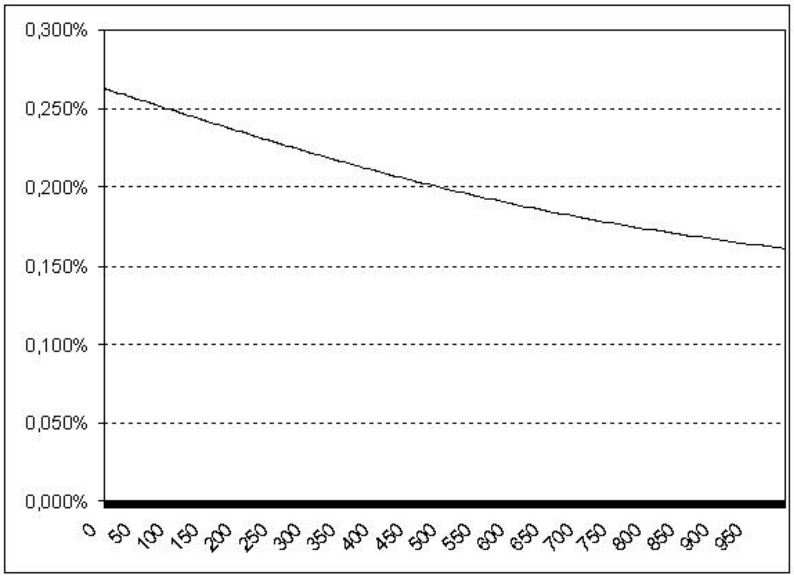

Figure 2: Yield curve example ($\sigma = 0.8$, $\sigma' = 1.5$, $r = 2\%$, $\delta = 0.1\%$)

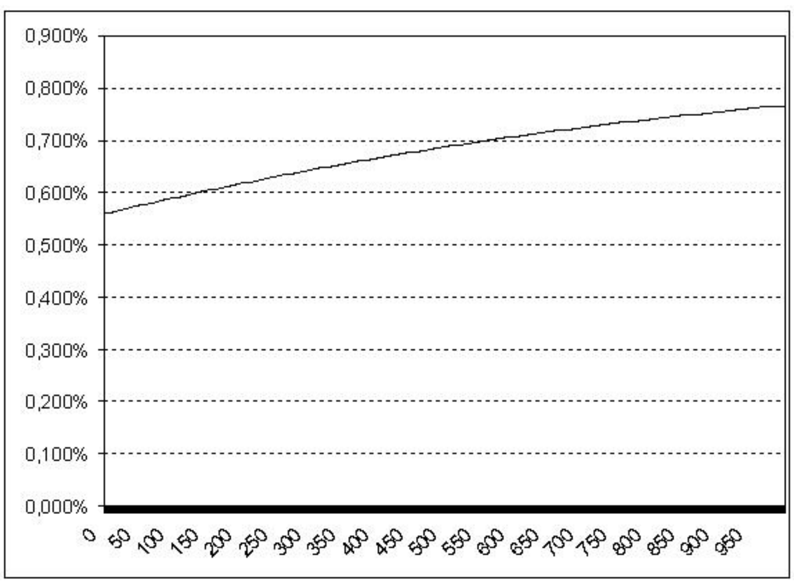

Figure 3: Yield curve example ($\sigma = 1.2$, $\sigma' = 1.5$, $r = 2\%$, $\delta = 0.1\%$)

An interesting fact is that ecological discount rates converge slowly to their asymptotic value. Another very interesting fact is that, when σ is low, the rate is low, but, even when σ is high, the environmental rate is still low in the medium run because the curve is increasing.

4 Uncertainty about the elasticity of substitution σ

We have seen that the two cases, $\sigma < 1$ and $\sigma > 1$ were really different. An interesting issue is therefore to understand what happens if we do not know in which case we are. To better understand the issue, we are going to consider two possible values for σ. More precisely, we consider that σ can be equal to $\sigma_l < 1$ with probability p or it can be equal to $\sigma_h > 1$ with probability $1-p$, p being in the interval $(0;1)$. To simplify the approach, we also consider that σ' is large enough so that $\sigma_l \sigma' > 1$.
We are going to show that, independently of $p > 0$, $B^*_\infty = \delta$. This is in accordance with the intuition that, asymptotically at least, the rate to be taken into account is the smallest.

To prove that, we are going to start again with the optimization problem:

$$\sum_{t=0}^{\infty} \exp(-\delta t)[pV(\sigma_l; x_t, \overline{y}) + (1-p)V(\sigma_h; x_t, \overline{y})]$$

$$s.t. \;\; a_{t+1} = \exp(r)a_t + w_t - x_t$$

Let's start with a result on the growth rate:

Proposition 12. *The asymptotic growth rate for the private good x^*_t does not depend on $p > 0$ and is equal to $g^*_\infty = \sigma_l(r-\delta)$*

This proposition means that, in terms of growth, everything is asymptotically as if p were equal to 1: the environmental issues dominate.

Now we can find the asymptotic ecological discount rate B^*_∞ in this random context:

Proposition 13. *The asymptotic ecological discount rate B^*_∞ does not depend on $p > 0$ and is equal to $B^*_\infty = \delta$*

An important conclusion is that the very possibility that the environmental issues dominate implies that, in the long run, everything happens as if the environmental issues do dominate. Obviously, this result is only asymptotic and based on the hypothesis that we never learn σ, an hypothesis that is quite restrictive. However, since we do not know σ and since we do not know when we would know it, this hypothesis seems to be an acceptable approximation of reality and the case $\sigma < 1$ is really relevant for that very reason.

5 Application to the expected environmental return on investment

Investing now to get benefits in the future is one of the central issues in finance. Here we develop a notion equivalent to the time value of money in an environmental framework. In other words, we investigate the fair increase in environmental good at time T to compensate, in utility terms, an investment at time 0. As before we are going to reason at the margin of an equilibrium trajectory.

Definition 3. *We denote by ω_t^* the marginal increase in environmental good at time $t+1$ to compensate a marginal investment at time t. This value is defined by:*

$$e^{-\omega_t^*} = e^{-\delta} \frac{\partial_2 V(x_{t+1}^*, \overline{y})}{\partial_1 V(x_t^*, \overline{y})}$$

The associated discount rate between periods 0 and T is defined by[11]*:*

$$e^{-\Omega^*(T)T} = e^{-\delta T} \frac{\partial_2 V(x_T^*, \overline{y})}{\partial_1 V(x_0^*, \overline{y})}$$

Now we can relate $\Omega^*(T)$ to the ecological discount rate and deduce asymptotic results for $\Omega^*(T)$.

Proposition 14. *$\Omega^*(T)$ is related to $B^*(T)$ by the following equation:*

$$\Omega^*(T) = B^*(T) + \frac{\sigma - 1}{\sigma^2 T} \ln(\rho_0^*)$$

It's now easy to deduce the asymptotic properties of $\Omega^*(T)$.

Proposition 15. *If we define Ω_∞^* by $\lim_{T\to\infty} \Omega^*(T)$ then:*

$$\Omega_\infty^* = B_\infty^*$$

This result must be noticed since it means, at least in the case where $\sigma < 1$, that there is no link between the long run interest rate R_∞^* and the long run return for investments in environmental projects, Ω_∞^*.

As before we can also construct yield curves $T \mapsto \Omega^*(T)$. From Proposition 14, we see that yield curves for the ecological discount rate differ from yield

[11] Here we cannot just sum the ω's because the two goods involved are different

curves for Ω by a purely economic term that vanishes in the long run. However, this term is important since it embeds the wealth of the whole economy. Indeed, this term $(\frac{\sigma-1}{\sigma^2 T}\ln(\rho_0^*))$ is decreasing in x_0^* (which is a good proxy for the total wealth of the economy without liquidity constraint) and turns out to be negative for large values of x_0^*. Therefore, the shape of the Ω yield curve factors in the total wealth of the economy, even though this dimension vanishes asymptotically.

The results of our simulations are presented below and we can see an hump-shaped yield curve that appears because of the complex combination of the wealth effect with the two traditional effects discussed above (depending on σ and σ').

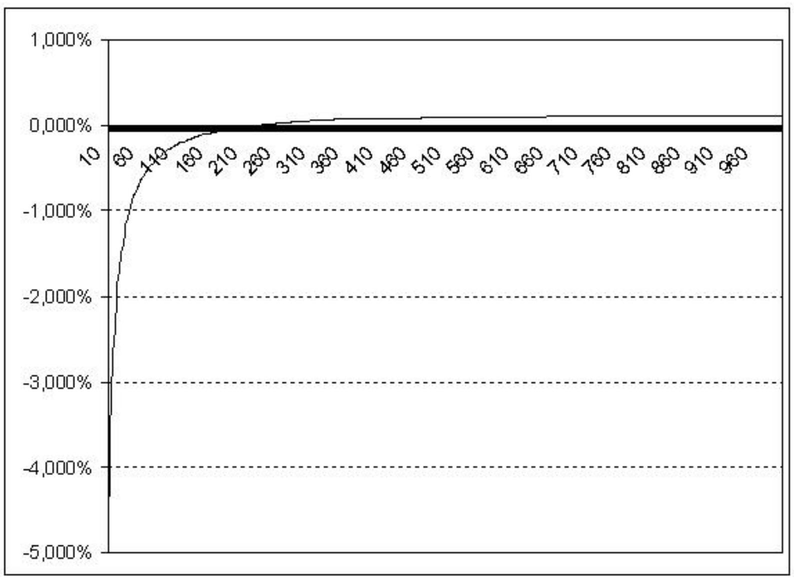

Figure 4: Yield Curve for Ω ($\sigma = 0.8$, $\sigma' = 1.5$, $r = 2\%$, $\delta = 0.1\%$, $\overline{y} \ll x_0^*$)

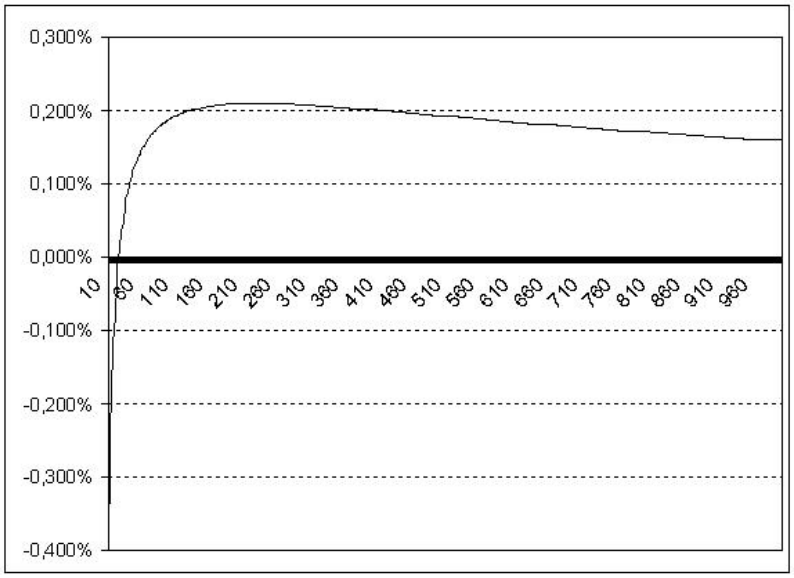

Figure 5: Yield Curve for Ω ($\sigma = 0.8$, $\sigma' = 1.5$, $r = 2\%$, $\delta = 0.1\%$, $\overline{y} \simeq x_0^*$)

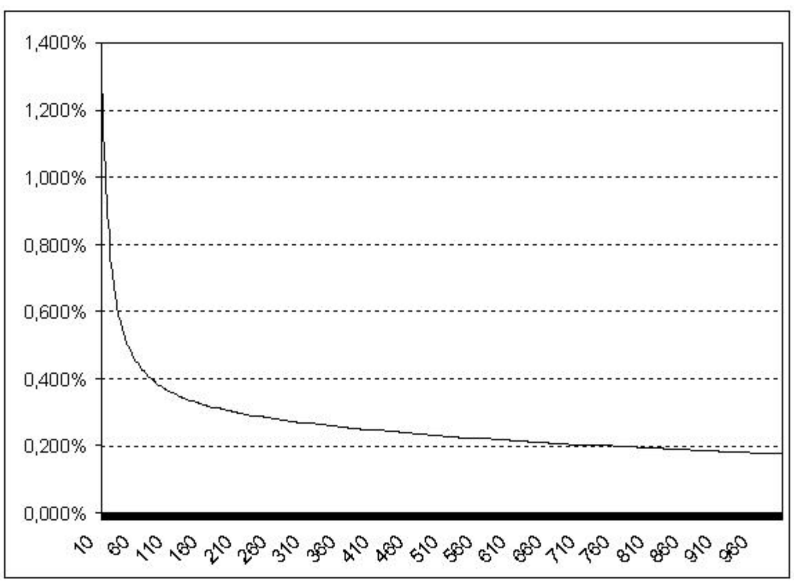

Figure 6: Yield Curve for Ω ($\sigma = 0.8$, $\sigma' = 1.5$, $r = 2\%$, $\delta = 0.1\%$, $\overline{y} \gg x_0^*$)

Without the wealth effect, the curves would be decreasing. However, the wealth effect can be of great importance in the short run and we see that a poor country is characterized by large rates in the short run (and therefore a decreasing curve at least for the first years) whereas for a rich country the rates can be negative during dozens of years.

Conclusion

Appendix: proofs

Proof of Proposition 1:

The implicit discount rate r_t^* for private goods between periods t and $t+1$ is uniquely defined by:

$$e^{-r_t^*} = e^{-\delta}\frac{\partial_1 V(x_{t+1}^*, \overline{y})}{\partial_1 V(x_t^*, \overline{y})} = e^{-\delta}\left(\frac{x_{t+1}^*}{x_t^*}\right)^{-\frac{1}{\sigma}}\left(\frac{{x_{t+1}^*}^{\frac{\sigma-1}{\sigma}} + \overline{y}^{\frac{\sigma-1}{\sigma}}}{{x_t^*}^{\frac{\sigma-1}{\sigma}} + \overline{y}^{\frac{\sigma-1}{\sigma}}}\right)^{\frac{1-\sigma\sigma'}{\sigma-1}}$$

Taking logarithms, this gives:

$$r_t^* = \delta + g_t^*/\sigma - \frac{1-\sigma\sigma'}{\sigma-1}\ln\left(\frac{{x_{t+1}^*}^{\frac{\sigma-1}{\sigma}} + \overline{y}^{\frac{\sigma-1}{\sigma}}}{{x_t^*}^{\frac{\sigma-1}{\sigma}} + \overline{y}^{\frac{\sigma-1}{\sigma}}}\right)$$

$$r_t^* = \delta + g_t^*/\sigma - \frac{1-\sigma\sigma'}{\sigma-1}\ln\left(\frac{1+{\rho_{t+1}^*}^{-1}}{1+{\rho_t^*}^{-1}}\right)$$

$$r_t^* = \delta + g_t^*/\sigma + \frac{1-\sigma\sigma'}{\sigma-1}\ln\left(\frac{1+{\rho_t^*}^{-1}}{1+{\rho_{t+1}^*}^{-1}}\right)$$

This is the second formula of Proposition 1. The first formula can be obtained by the same reasoning if we go back to:

$$r_t^* = \delta + g_t^*/\sigma - \frac{1-\sigma\sigma'}{\sigma-1}\ln\left(\frac{{x_{t+1}^*}^{\frac{\sigma-1}{\sigma}} + \overline{y}^{\frac{\sigma-1}{\sigma}}}{{x_t^*}^{\frac{\sigma-1}{\sigma}} + \overline{y}^{\frac{\sigma-1}{\sigma}}}\right)$$

$$r_t^* = \delta + g_t^*/\sigma - \frac{1-\sigma\sigma'}{\sigma-1}\ln\left[\left(\frac{x_{t+1}^*}{x_t^*}\right)^{\frac{\sigma-1}{\sigma}}\frac{1+\left(\frac{\overline{y}}{x_{t+1}}\right)^{\frac{\sigma-1}{\sigma}}}{1+\left(\frac{\overline{y}}{x_{t+1}}\right)^{\frac{\sigma-1}{\sigma}}}\right]$$

$$r_t^* = \delta + g_t^*/\sigma - \frac{1-\sigma\sigma'}{\sigma-1}\frac{\sigma-1}{\sigma}g_t^* - \frac{1-\sigma\sigma'}{\sigma-1}\ln\left(\frac{1+\rho_{t+1}^*}{1+\rho_t^*}\right)$$

$$r_t^* = \delta + g_t^*\sigma' + \frac{1-\sigma\sigma'}{\sigma-1}\ln\left(\frac{1+\rho_t^*}{1+\rho_{t+1}^*}\right)$$

This formulation will be useful when $\sigma > 1$ whereas the other one will be useful for $\sigma < 1$. $\square$

Proof of Proposition 2:

We remind that $\frac{\partial_2 V}{\partial_1 V} = \left(\frac{x}{y}\right)^{1/\sigma}$.

Therefore, $e^{-\beta_t^*} = e^{-\delta} \frac{(\partial_2 V)_{t+1}}{(\partial_2 V)_t} = e^{-\delta} \frac{(\partial_1 V)_{t+1}}{(\partial_1 V)_t} \left(\frac{x_{t+1}}{x_t^*}\right)^{1/\sigma} = e^{-r_t^*} e^{g_t^*/\sigma}$

Hence, $\beta_t^* = r_t^* - g_t^*/\sigma$ and this proves Proposition 2. $\square$

Proof of Proposition 3:

We have the formula of Proposition 1:

$$r_t^* = \delta + g_t^* \sigma' + \frac{1 - \sigma\sigma'}{\sigma - 1} \ln \left(\frac{1 + \rho_t^*}{1 + \rho_{t+1}^*} \right)$$

We note that when σ is greater than one, as soon as g_t^* has a lower bound strictly greater than zero, ρ_t tends to zero. In this case, it is straightforward to see that:

$$r_t^* \to \delta + g_t^* \sigma'$$

It's now easy to conclude that $R^*(T) \to \delta + g^* \sigma'$ using Césàro's theorem.

For the long run ecological discount rate, we use Proposition 2 to conclude that $B^*(T) - R^*(T) \to g^*/\sigma$ and this leads to the result:

$$B^*(T) \to \delta + g^*(\sigma' - 1/\sigma)$$

In the $\sigma < 1$ case we come back to the other part of Proposition 1:

$$r_t^* = \delta + g_t^*/\sigma + \frac{1 - \sigma\sigma'}{\sigma - 1} \ln \left(\frac{1 + \rho_t^{*-1}}{1 + \rho_{t+1}^{*}{}^{-1}} \right)$$

Here, however, in the long run, $\rho_t^* \to +\infty$ so that we have directly:

$$r_t^* \to \delta + g^*/\sigma$$

Using Césàro's theorem we get $R^*(T) \to \delta + g^*/\sigma$ and with the help of Proposition 2 we have the result on B that is $B^*(T) \to \delta$ $\square$

Proof of Proposition 7:

The first thing to note is that $y_t^* = \overline{y}$.

Then, since we supposed that r is greater than δ we have $\exp(r-\delta) > 1$ so that λ_t and $\partial_x V(x_t^*, \overline{y})$ are both decreasing and tend to zero. The natural consequence is that the consumption of the private good x_t^* grows and tends to $+\infty$.

The growth path x_t^* is then characterized by:

$$\partial_x V(x_t^*, \overline{y}) = \lambda_t = \frac{\lambda_0}{\exp((r-\delta)t)}$$

Therefore,

$$x_t^{*-\frac{1}{\sigma}} \left[x_t^{*\frac{\sigma-1}{\sigma}} + \overline{y}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{1-\sigma\sigma'}{\sigma-1}} = \frac{\lambda_0}{\exp((r-\delta)t)}$$

As we did in the preceding parts we are going to consider two cases depending on σ being larger or smaller than 1.

- The $\sigma > 1$ case:

$$x_t^{*-\sigma'} \sim_\infty \frac{\lambda_0}{\exp((r-\delta)t)}$$

$$\ln(\frac{x_{t+1}^*}{x_t^*}) \sim_\infty \frac{r-\delta}{\sigma'}$$

  Hence, the asymptotic growth rate is the same as if there were no consideration of the environmental good:

$$g_\infty^* = \frac{r-\delta}{\sigma'}$$

- The $\sigma < 1$ case:

$$x_t^{*-\frac{1}{\sigma}} \sim_\infty \frac{\lambda_0}{\overline{y}^{\frac{1}{\sigma}-\sigma'} \exp((r-\delta)t)}$$

$$\ln(\frac{x_{t+1}^*}{x_t^*}) \sim_\infty \sigma(r-\delta)$$

  Hence, the growth rate in that case is given by:

$$g_\infty^* = \sigma(r-\delta)$$

As before, it is also possible to consider $\sigma = 1$ by taking a Cobb-Douglas function for V: $V(x_t, y_t) = \frac{(x_t y_t)^{\frac{1-\sigma'}{2}}}{1-\sigma'}$ and we eventually obtain $g_\infty^* = \frac{2(r-\rho)}{1+\sigma'}$ □

Proof of Proposition 9: *(This proof can be omitted at first reading)*

The first thing to do is to write the result of Proposition 2 and to deduce a useful expression for $B^*(T)$.

We have:

$$\beta_t = r - \frac{g_t^*}{\sigma}$$

$$\Rightarrow B^*(T) = r - \frac{1}{\sigma}\frac{1}{T}\sum_{t=0}^{T-1} g_t^*$$

$$\Rightarrow B^*(T) = r - \frac{1}{\sigma}\frac{1}{T}\ln\left(\frac{x_T^*}{x_0^*}\right)$$

Therefore, the only thing to prove is that $\forall t, x_t^*$ is a continuous function of σ. But we know that the growth path is defined by the first order condition $\partial_x V(x_t^*;\sigma) = \frac{\lambda_0}{\exp((r-\delta)t)}$[12] where we omitted the reference to $\overline{y}$ here since we focus on σ. Then it is easy to see that the only two things we need to prove are:

- The Lagrange multiplier λ_0 is a continuous function of σ.
- The function $g(\xi,\sigma)$ implicitly defined by $\partial_1 V(g(\xi,\sigma);\sigma) = \xi$ is continuous.

The second point is easy. Notice first that the function $(x,\sigma) \mapsto V(x;\sigma)$ can be extended to a C^2 function (the proof is easy). Then, by the implicit function theorem, $g(\xi,\sigma)$ is a C^1 function $((\xi,\sigma) \in (\mathbb{R}^{+*})^2)$.
Therefore, the only thing to prove is that the first Lagrange multiplier λ_0 is a continuous function of σ. Let us recall that λ_0 is defined by the resources constraint:

$$\sum_{t=0}^{\infty} x_t^* e^{-rt} = a_0 + \sum_{t=0}^{\infty} w_t e^{-rt} (:= \Lambda_\infty{}^{13})$$

$$\sum_{t=0}^{\infty} g(\lambda_0 \exp((\delta - r)t), \sigma) e^{-rt} = \Lambda_\infty$$

Here, we cannot apply directly the implicit function theorem to the left hand side. However, if we consider the restricted optimization problem with a fixed time horizon T[14] then the associated Lagrange multiplier (λ_0^T) is implicitly defined by

$$\sum_{t=0}^{T} g(\lambda_0^T \exp((\delta - r)t), \sigma) e^{-rt} = a_0 + \sum_{t=0}^{T} w_t e^{-rt} (:= \Lambda_T)$$

[12] It's very important here to consider $v(x,y) = \left[\frac{1}{2}x^{\frac{\sigma-1}{\sigma}} + \frac{1}{2}y^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$ with the weights $\frac{1}{2}$ to extend the function properly. Obviously, it doesn't change anything to our preceding results since it is only a multiplicative scalar adjustment

[13] This quantity is supposed finite for the problem to have a solution.

[14] $Max \sum_{t=0}^{T} \exp(-\rho t) u(x_t, \overline{y})$ s.t. $a_{t+1} = \exp(r)a_t + w_t - x_t$

and the implicit function theorem applies: λ_0^T is a C^1 function of σ.

Now, we can approximate λ_0 by λ_0^T and this gives:
$|\lambda_0(\sigma) - \lambda_0(\tilde{\sigma})| \leq |\lambda_0(\sigma) - \lambda_0^T(\sigma)| + |\lambda_0^T(\sigma) - \lambda_0^T(\tilde{\sigma})| + |\lambda_0^T(\tilde{\sigma}) - \lambda_0(\tilde{\sigma})|$.
Hence, we see that the only thing to prove is a pointwise convergence in the sense that, for σ fixed, we have a convergence of $\lambda_0^T(\sigma)$ towards $\lambda_0(\sigma)$ as $T \to \infty$.
To prove that let's introduce $F_T : z \mapsto \sum_{t=0}^{T} g(z\exp((\delta - r)t), \sigma)e^{-rt}$ and similarly $F : z \mapsto \sum_{t=0}^{\infty} g(z\exp((\delta - r)t), \sigma)e^{-rt}$. These two functions are positive and decreasing because g is a positive and decreasing function of ξ. Moreover, F_T is continuous and there is a pointwise convergence of F_T towards F. By monotony, F_T converges towards F uniformly on every compact set and therefore, F is a continuous function and so is the inverse of the function F.
By the second Dini's theorem then, the inverse of the function F_T converges uniformly on every compact set towards the inverse of the function F.
But $\lambda_0^T - \lambda_0 = F_T^{-1}(\Lambda_T) - F^{-1}(\Lambda_\infty)$ and hence, since $\Lambda_T \to \Lambda_\infty$, we are done with the proof. □

Proof of Proposition 10:

Let us go back the the first order conditions that define the growth path.
We have:

$$\partial_x V(x_t^*, \overline{y}) = e^{r-\delta}\partial_x V(x_t^* e^{g_t^*}, \overline{y})$$

Therefore, g, as a function of x is defined implicitly by (we now omit the $\overline{y}$ terms):

$$V'(x) = e^{r-\delta}V'\left(xe^{g(x)}\right)$$

If $\sigma = 1$, we are dealing with Cobb-Douglas functions and then the growth rate is clearly independent of x and the result is proved.

Otherwise, since x_t^* is an increasing sequence, the variation properties of g_t^* are given by the sign of $g'(x)$ that is going to be computed now.

Taking logs and deriving we get:

$$\frac{V''(x)}{V'(x)} = \frac{V''(xe^{g(x)})}{V'(xe^{g(x)})}e^{g(x)}(1 + g'(x)x)$$

Hence, the sign of $g'(x)$ is the sign of $V'(x)V''(xe^{g(x)})e^{g(x)} - V'(xe^{g(x)})V''(x)$.
This sign is simply the sign of $\frac{d}{dx}\frac{V'(xe^g)}{V'(x)}$ where g is now an independent variable.

The latter expression can be written as:

$$e^{-g/\sigma}\frac{d}{dx}\left[\frac{\overline{y}+(xe^{g})^{\frac{\sigma}{\sigma-1}}}{\overline{y}+x^{\frac{\sigma}{\sigma-1}}}\right]^{\frac{1-\sigma\sigma'}{\sigma-1}}$$

The sign of this derivative is the sign of:

$$\frac{1-\sigma\sigma'}{\sigma-1}\frac{\sigma-1}{\sigma}\left(e^{g\frac{\sigma-1}{\sigma}}-1\right)=\frac{1-\sigma\sigma'}{\sigma}\left(e^{g\frac{\sigma-1}{\sigma}}-1\right)$$

Since $g>0$ in our context, this expression has the same sign as the product $(1-\sigma\sigma')(\sigma-1)$ and this proves our result. □

Proof of Proposition 12:

Using the same notations as in Proposition 7 we know that the growth path x_t^* is characterized by:

$$p\partial_x V(\sigma_l;x_t^*,\overline{y})+(1-p)\partial_x V(\sigma_h;x_t^*,\overline{y})=\frac{\lambda_0}{\exp((r-\delta)t)}$$

Therefore,

$$px_t^{*-\frac{1}{\sigma_l}}\left[x_t^{*\frac{\sigma_l-1}{\sigma_l}}+\overline{y}^{\frac{\sigma_l-1}{\sigma_l}}\right]^{\frac{1-\sigma_l\sigma'}{\sigma_l-1}}+(1-p)x_t^{*-\frac{1}{\sigma_h}}\left[x_t^{*\frac{\sigma_h-1}{\sigma_h}}+\overline{y}^{\frac{\sigma_h-1}{\sigma_h}}\right]^{\frac{1-\sigma_h\sigma'}{\sigma_h-1}}=\frac{\lambda_0}{\exp((r-\delta)t)}$$

Using the asymptotic results derived earlier in Proposition 7, we get that:

$$px_t^{*-\frac{1}{\sigma_l}}+o\left(x_t^{*-\frac{1}{\sigma_l}}\right)+(1-p)x_t^{*-\sigma'}+o\left(x_t^{*-\sigma'}\right)=\frac{\lambda_0}{\exp((r-\delta)t)}$$

But, since we supposed that $\sigma_l\sigma'>1$, we have:

$$px_t^{*-\frac{1}{\sigma_l}}\sim_\infty\frac{\lambda_0}{\exp((r-\delta)t)}$$

Hence, as in Proposition 7, we get the asymptotic growth rate which is here $g_\infty^*=\sigma_l(r-\delta)$. □

Proof of Proposition 13:

Let's recall first the definition of $B^*(T)$ in this context:

$$B^*(T)=\delta-\frac{1}{T}\ln\left[\frac{p\partial_y V(\sigma_l;x_T,\overline{y})+(1-p)\partial_y V(\sigma_h;x_T,\overline{y})}{p\partial_y V(\sigma_l;x_0,\overline{y})+(1-p)\partial_y V(\sigma_h;x_0,\overline{y})}\right]$$

To prove our result, it is sufficient to prove that the expression in the logarithm remains bounded as T increases. Hence, we are going to prove that the following expression is bounded:

$$p\overline{y}^{-\frac{1}{\sigma_l}}\left[x_T^{*\frac{\sigma_l-1}{\sigma_l}}+\overline{y}^{\frac{\sigma_l-1}{\sigma_l}}\right]^{\frac{1-\sigma_l\sigma'}{\sigma_l-1}}+(1-p)\overline{y}^{-\frac{1}{\sigma_h}}\left[x_T^{*\frac{\sigma_h-1}{\sigma_h}}+\overline{y}^{\frac{\sigma_h-1}{\sigma_h}}\right]^{\frac{1-\sigma_h\sigma'}{\sigma_h-1}}$$

The first part of the expression converges toward $p\overline{y}^{-\sigma'}$ and is therefore bounded.
For the second part of the expression, $x_T^{*\frac{\sigma_h-1}{\sigma_h}}+\overline{y}^{\frac{\sigma_h-1}{\sigma_h}}\rightarrow\infty$ so that, since $\frac{1-\sigma_h\sigma'}{\sigma_h-1}<0$, the second part of the expression tends toward 0 and this proves the result. □

Proof of Proposition 14:

By definition:

$$e^{-\Omega^*(T)T}=e^{-\delta T}\frac{\partial_2 V(x_T,\overline{y})}{\partial_1 V(x_0^*,\overline{y})}=e^{-B^*(T)T}\frac{\partial_2 V(x_0,\overline{y})}{\partial_1 V(x_0^*,\overline{y})}=e^{-B^*(T)T}\left(\frac{\overline{y}}{x_0^*}\right)^{-\frac{1}{\sigma}}$$

To conclude, we just need to take logarithms of both sides. □

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