Socially efficient discounting under ambiguity aversion

Christian Gollier
Toulouse School of Economics (LERNA and EIF)

Johannes Gierlinger
Toulouse School of Economics (LERNA)

June 6, 2007
Abstract

We consider an economy with an ambiguity-averse representative agent who faces an uncertain growth of her consumption. We show that it is not true in general that ambiguity aversion induces the representative agent to be more willing to save. In other words, ambiguity aversion does not necessarily reduce the equilibrium interest rate. We show that ambiguity aversion has a wealth effect and a pessimistic effect on the equilibrium interest rate. Under decreasing ambiguity aversion, ambiguity aversion has an effect equivalent to reduce the expected future wealth, thereby reducing the interest rate. It has also the pessimistic effect to raise the probability of the worst scenarios, which generates in general an additional reduction of the interest rate.

Keywords: Discount rate, prudence, smooth ambiguity aversion, Ramsey rule, pessimism.
1 Introduction

The emergence of public policy problems associated with the sustainability of our economic growth has raised a considerable interest for the determination of a socially efficient discount rate. This debate has recently culminated with the publication of two reports. On one side, the Copenhagen Consensus (Lomborg (2004)) put top priority to public programs yielding immediate benefits (fighting malaria and AIDS, improving water supply, ...), and rejected the idea to invest much in the prevention of global warming. On the other side, the Stern Review (Stern (2007)) put a tremendous pressure for acting quickly and heavily against global warming.

Because global warming will really affect our economies in more than 100 years, the choice of the rate at which these costs are discounted plays a key role in the conclusion. While Stern uses 1.4% per year, the Copenhagen Consensus applies a rate of 5% for costs and benefits of climate mitigation. For the sake of illustrating the power of discounting, consider a project which yields its benefits in $t$ years time. For a horizon $t = 100$ the Copenhagen Consensus would require a rate-of-return already 36 times higher than Stern. This disagreement becomes even more pronounced if $t = 200$, where critical benefits are 1300 times higher for the Copenhagen Consensus.

Yet, despite their disagreement on parameter values\(^1\), both aforementioned projects agree on the methods to determine the social discount rate. In particular, they consider an economy where the stochastic process which generates tomorrow’s wealth is perfectly known.

However, in view of the fact that typically predictions of economic growth within, say, 3 years time might already conflict, this assumption appears stark on the scale of decades, all the more when considering centuries. Several authors pointed out that structural parameter uncertainty might indeed be more important than risk for long time horizons. Both Weitzman (2007a) and Gollier (2007) show that as soon as one accepts uncertainty, the far distant future should be discounted at a smaller rate. Hence, whether the appropriate rate is closer to 1.4% than to 5%, not only depends on the

\(^1\)For instance, the projects mentioned above disagree on the rate of pure preference for the presence, i.e. how strongly, if at all, the well-being of future generations should be discounted. While the Copenhagen Consensus applies a rate of 1.5%, which is close to real world estimates of individuals, the Stern review advocates a rate of only 0.1%, based on ethical grounds. See Weitzman (2007b) for a discussion.
preferences of the representative agent, but also on the model-uncertainty and the time horizon at hand.

Whereas these two papers are based on a completely standard expected utility modeling, we are motivated by the question, how to discount cash flows if the representative agent is ambiguity-averse. In the present context, ambiguity manifests itself in not knowing from which of several candidate distributions future consumption will be drawn. Following the pioneering work by Ellsberg (1961), ample evidence in favor of ambiguity aversion has been accrued\(^2\). It suggests that human beings violate Savage’s “Sure Thing Principle”, in the sense that they dislike mean-preserving spreads over priors. In general, ambiguous consequences are not evaluated according to their arithmetic mean. Hence, a natural question to ask is, whether a subjective utility model systematically under- (or over-) estimates the social discount rate.

The present paper will investigate whether, given ambiguity-aversion, standard subjective utility impose a bias. Thanks to using “smooth ambiguity preferences” as proposed by Klibanoff, Marinacci and Mukerji (2005, 2007) we are able to employ well-known results from risk-theory for our purposes. In the subsequent sections we will identify joint requirements on tastes and uncertainty such that the discount rate is indeed lower if ambiguity preferences are taken into account.

We will first explain, why in general, ambiguity aversion may also increase the socially efficient discount rate. This is connected to two, possibly opposing, effects of ambiguity aversion on marginal utilities. On the one hand, an implicit *pessimistic effect* acts as if weights were shifted towards unfavorable distributions, while at the same time, an implicit *wealth effect* shifts the level of marginal expected utilities. We then go on to introduce the notion of *non increasing absolute ambiguity aversion* (DAAA), under which the wealth effect on savings is positive. Finally, we will derive pairs of conditions on the standard utility function and the stochastic ordering of candidate distributions which guarantee that indeed, under DAAA, the socially efficient discount rate is lower than in the subjective utility maximizer benchmark.

---

\(^2\)The Ellsberg-Paradox (Ellsberg (1961)) refers to the outcome of the following experiment. Participants faced the choice to place a fair bet on the draw from either Urn 1 or Urn 2. Urn 1 contained 50 red balls and 50 black balls, Urn 2 contained 100 balls with unknown proportions of red vs black. A large majority chose to draw from Urn 1, even though expected payoffs were obviously equivalent.
The effect of parameter uncertainty on discount rates has been studied in several papers. First and foremost, Weitzman (1998, 2001) and among others Groom et. al. (2004) examine the question, how misspecifications affect the efficient “certainty-equivalent” discount rate. Weitzman shows that asymptotically, for a large time horizon, the “certainty-equivalent” rate approaches the lowest possible one. However, none of the above derived equilibrium rates endogenously. In contrast, Weitzman (2007a) and Gollier (2007) determine discount rate in a partial equilibrium model. In particular, like in the present paper, they consider uncertainty on future economic growth. They reach similar conclusions, namely that discount rates should be decreasing over time. However, since we do not focus on the impact of ambiguity on the term structure of discount rates, one can relate our comparative statics results to any consumption-based models like Gollier (2002a, 2002b), who abstracts from model uncertainty. Finally, our paper is technically related to Gollier (2006). In a finance context, he investigates comparative statics results of an increase in ambiguity aversion on the demand for risky assets. He shows that, in general, omitting ambiguity aversion cannot be corrected for by assuming a higher degree of risk aversion.

The remainder of the paper is organized as follows. Section 2 introduces the basic model and presents the equilibrium pricing formula. In Section 3 an analytical example yields an adapted Ramsey-rule for the interest rate under ambiguity. Before deriving our main results in Section 5, we establish non increasing absolute ambiguity aversion. Section 6 concludes.

2 The model

We consider an economy à la Lucas (1978). Each agent in the economy is endowed with a tree which produces $\tilde{c}_t$ fruits at date $t$, $t = 0, 1, 2, ...$. There is a market for zero-coupon bonds at date 0 in which agents may exchange the delivery of one fruit today against the delivery of $e^{rt}$ fruits for sure at date $t$. Thus, the real interest rate associated to maturity $t$ is $r_t$.

The distribution of $\tilde{c}_t$ is a function of a parameter $\theta$ that can take value $1, 2, ..., n$ with probability $q_1, ..., q_n$, respectively. The cumulative distribution function of $\tilde{c}_t$ conditional to $\theta$ is denoted $F_{t\theta}$. The random variable with such a cumulative distribution function is denoted $\tilde{c}_{t\theta}$. An ambiguous environment for $\tilde{c}_t$ is thus fully described by vector $C = (q_1, \tilde{c}_{t1}; ...; q_n, \tilde{c}_{tn})$. 
Following Klibanoff, Marinacci and Mukerji (2005) and its recursive generalization (Klibanoff, Marinacci and Mukerji (2007)), we assume that the preferences of the representative agent exhibit smooth ambiguity aversion. For each plausible probability distribution $F_\theta$, the agent who purchased $\alpha$ zero-coupon bonds associated to date $t$ computes her future expected utility $U_t(\alpha, \theta) = \mathbb{E}_u(\tilde{c}_t\theta + \alpha e^{rt}) = \int u(c + \alpha e^{rt})dF_\theta(c)$ conditional to $F_\theta$ being the true distribution. We assume that $u$ is twice differentiable, increasing and concave, so that $U(\cdot, \theta)$ is concave in the investment $\alpha$, for all $\theta$. Ex ante, for a given investment $\alpha$, the welfare of the agent is measured by $V_t(\alpha)$ with

$$
\phi(V_t(\alpha)) = \sum_{\theta=1}^{n} q_\theta \phi(U_t(\alpha, \theta)) = \sum_{\theta=1}^{n} q_\theta \phi \left( \mathbb{E}_u(\tilde{c}_t\theta + \alpha e^{rt}) \right), \tag{1}
$$

The shape of $\phi$ describes the investor’s attitude towards ambiguity (or parameter uncertainty). Function $\phi$ is assumed to be three times differentiable, increasing and concave. $V_t(\alpha)$ can be interpreted as the certainty equivalent of the uncertain conditional expected utility $U_t(\alpha, \tilde{\theta})$. A linear $\phi$ means that the investor is neutral to ambiguity. In such a case, the DM is indifferent to any mean-preserving spread of $U_t(\alpha, \tilde{\theta})$, and $V_t(\alpha)$ can be represented by a subjective expected utility functional $V_{tSEU}^\alpha(c) = \mathbb{E}_u(\tilde{c}_t + \alpha e^{rt})$, where $\tilde{c}_t$ is the random variable that is distributed as $(\tilde{c}_{t1}, q_1; ...; \tilde{c}_{tn}, q_n)$. On the contrary, a concave $\phi$ is synonymous of ambiguity aversion in the sense that the DM dislikes any mean-preserving spread of the conditional expected utility $U_t(\alpha, \tilde{\theta})$.

An interesting particular case arises when the absolute ambiguity aversion $A(U) = -\phi''(U)/\phi'(U)$ is constant, so that $\phi(U) = -A^{-1} \exp(-AU)$. As proved by Klibanoff, Marinacci and Mukerji (2005), the ex ante welfare $V(\alpha)$ tends to maxmin expected utility functional $V_{tMEU}^\alpha(c) = \min_{\theta} \mathbb{E}_u(\tilde{c}_t\theta + \alpha e^{rt})$ when the degree of absolute ambiguity aversion $\phi$ tends to infinity. Thus, the Gilboa and Schmeidler (1989)’s maxmin criteria is a special case of this model.

The optimal investment $\alpha^*$ maximizes the intertemporal welfare of the investor, which is written as:

$$
\alpha^* \in \arg \max_{\alpha} \quad u(c_0 - \alpha) + e^{-\delta t} V_t(\alpha). \tag{2}
$$

Parameter $\delta$ is the rate of pure preference for the present. If $\phi$ and $u$ are strictly concave, the objective function is concave in $\alpha$ and the solution to
program (2), when it exists, is unique. The necessary and sufficient condition of program (2) is written as
\[ u'(c_0 - \alpha^*) = e^{-st}V_t''(\alpha^*) \]

Fully differentiating equation (1) with respect to \( V \) yields
\[ V_t'(\alpha) = e^{r_t} \sum_{\theta=1}^{n} q_{\theta} \phi'(E u(\tilde{c}_t) + \alpha e^{r_t}) \frac{E u'(\tilde{c}_t)}{\phi'(V_t(\alpha))}. \]

Because we assume that all agents have the same preferences and the same stochastic endowment, the equilibrium condition on the market for the zero-coupon bond associated to maturity \( t \) is \( \alpha^* = 0 \). Combining the above two equations implies the following equilibrium condition:
\[ r_t = \delta - \frac{1}{t} \ln \left[ \frac{\sum_{\theta=1}^{n} q_{\theta} \phi'(E u(\tilde{c}_t)) E u'(\tilde{c}_t)}{u'(c_0)} \right]. \] (3)

This is also the socially efficient rate at which sure benefits and costs occurring at date \( t \) must be discounted in any cost-benefit analyses at date 0.

As a benchmark, consider the case of an ambiguity neutral representative agent. In that case, we get the standard bond pricing formula \( r_t = \delta - t^{-1} \ln [E u'(\tilde{c}_t)/u'(c_0)].^3 \) In this special case, we see that the riskiness of future consumption reduces the socially efficient discount rate if and only if \( u' \) is convex, i.e., if the representative agent is prudent (Leland (1968), Drèze and Modigliani (1972), Kimball (1990)).

Our goal in this paper is to determine the conditions under which ambiguity and ambiguity aversion reduce the discount rate. An ambiguous environment \( C = (q_1, \tilde{c}_1; \ldots; q_n, \tilde{c}_n) \) is said to be acceptable if the supports of the \( \tilde{c}_{t\theta} \) are in the domain of \( u \), and if all \( E u'(\tilde{c}_{t\theta}) \) are in the domain of \( \phi \). The set of acceptable ambiguous environment is denoted \( \Psi \).

3 An analytical solution

Let us consider the following specification:

\(^3\text{See for example Cochrane (2001).}\)
• The plausible distributions of $\ln \tilde{c}_{t \theta}$ are all normally distributed with the same variance $\sigma^2 t$, and with mean $\ln c_0 + \theta t$.\(^4\)

• The prior distribution on $\theta$ is normally distributed with mean $\mu$ and variance $\sigma^2_0$.\(^5\)

• The representative agent’s preferences exhibit constant relative risk aversion $\gamma = -cu''(c)/u'(c)$, i.e., $u(c) = c^{1-\gamma}/(1 - \gamma)$.

• The representative agent’s preferences exhibit constant relative ambiguity aversion $\eta = -|u|\phi''(u)/\phi'(u) \geq 0$. This means that $\phi(U) = h(kU)^{1-\eta/k}$, where $k = \text{sign}(1 - \gamma)$ is the sign of $u$.

We show in the Appendix that there is an analytical solution for the discount rate, which is given in this case by:

$$r_t = \delta + \gamma \mu - \frac{1}{2} \gamma^2 (\sigma^2 + \sigma^2_0 t) - \frac{1}{2} \eta \frac{1 - \gamma^2}{\text{sign}(1 - \gamma)} \sigma^2_0 t.$$  \(^{(4)}\)

The first two terms in the right-hand side of this equation correspond to the classical Ramsey rule. The interest rate is increasing in the expected growth rate of consumption $\mu$. When $\mu$ is positive, decreasing marginal utility implies that the marginal utility of consumption is expected to be smaller in the future than it is today. This yields a positive interest rate. The third term expresses prudence. Because the riskiness of future consumption increases the expected marginal utility $Eu'(\tilde{c}_t)$ under prudence,\(^6\) this has a negative impact on the discount rate. Notice that the variance of consumption at date $t$ equals $\sigma^2 t + \sigma^2_0 t^2$, so that its increases at an increasing rate with respect to the time horizon. There, the precautionary effect has a relatively larger impact on the discount rate for longer horizons. This argument has been developed in Weitzman (2007a) and Gollier (2007) to justify a decreasing discount rate in an expected utility framework.

---

\(^4\)In continuous time, this would mean that the consumption process is a geometric brownian motion $d \ln c = \theta dt + \sigma dw$.

\(^5\)We consider the natural continuous extension of our model with a discrete distribution for $\theta$.

\(^6\)An agent is prudent if adding risk to future incomes raises the precautionary saving (Kimball (1990)). Leland (1968) has shown that an agent is prudent if and only if her marginal utility is a convex function of consumption. Power utility functions exhibit prudence.
The last term in the right-hand side of equation (4) characterizes the effect of ambiguity. Observe that it always tends to reduce the discount rate under positive ambiguity aversion ($\eta > 0$). This effect is increasing in the degree of ambiguity aversion $\eta$, in the degree of uncertainty $\sigma_0$, and in the time horizon $t$.

4 The case of risk neutrality

In this section, we examine the case of risk neutrality, $u(c) = c$. Let $\bar{c}_\theta = E\bar{c}_t\theta$ denote the expected consumption in date $t$ conditional to $\theta$. In that case, the pricing formula (3) is rewritten as

$$r_t = \delta - \frac{1}{t} \ln \left[ \sum_{\theta=1}^{n} q_\theta \phi'(\bar{c}_\theta) \right],$$

with $\phi(V_t(0)) = \sum_\theta q_\theta \phi(\bar{c}_\theta)$. Under ambiguity neutrality ($\phi'' \equiv 0$), we would have $r_t = \delta$. The following Lemma is useful for future results.\footnote{See Gollier and Kimball (1996) and Gollier (2001, section 2.5)}

**Lemma 1** Consider a three times differentiable increasing function $g : B \subset \mathbb{R} \rightarrow \mathbb{R}$. The following two conditions are equivalent:

1. For all random variables $\bar{x}$ whose support is in $B$, and any scalar $x_0$, $E g(\bar{x}) = g(x_0)$ implies $E g'(\bar{x}) \geq g'(x_0)$.

2. $g$ is such that $-g''(x)/g'(x)$ is nonincreasing in $x$.

**Proof:** Let us observe that condition 2 means that $-g''/g'$ be uniformly larger than $-g''/g'$, or that $-g'$ is a concave transformation of $g$. Thus, this means that there exists a concave function $f$ such that $-g'(x) = f(g(x))$ for all $x$.

Let us first prove that condition 1 holds for all random variables $\bar{x}$ if $g$ satisfies condition 2. Indeed, because we have that $-g'$ is a concave function $f$ of $g$, we have that

$$-E g'(\bar{x}) = Ef(g(\bar{x})) \leq f(E g(\bar{x})) = f(g(x_0)) = -g'(V)$$

where we used Jensen’s inequality. Thus, we have proved $2 \implies 1$. 

---
To prove $1 \Rightarrow 2$, let us assume by contradiction that $-g' = f(g)$ is not a concave transformation of $g$. This means that function $f$ is locally convex in an interval $[g(x_a), g(x_b)] \subset B$. Then take any distribution of $\tilde{x}$ with a support in in $[x_a, x_b]$. Then, it implies by the Jensen inequality that

$$-Eg'(\tilde{x}) = Ef(g(\tilde{x})) \geq f(E(g(\tilde{x}))) = f(g(x_0)) = -g'(V)$$

This contradicts condition 1. ■

Applying this lemma for $g = \phi$ and $\tilde{x}$ being distributed as $(q_1, \tau_1; \ldots; q_n, \tau_n)$ directly yields the following result.

**Proposition 1** Suppose that the representative agent is risk neutral. The socially efficient discount rate is smaller than under ambiguity neutrality for all acceptable ambiguous environments $C \in \Psi$ if and only if $\phi$ exhibits non-increasing absolute ambiguity aversion, i.e., iff $A'(U) = (-\phi''(U)/\phi'(U))' \leq 0$ for all $U$.

Under risk neutrality, the driving force for the impact of ambiguity on the interest rate is not ambiguity aversion itself, but whether the degree of ambiguity aversion is increasing or decreasing with the level of expected utility $U$. In particular, Proposition 1 tells us that, under risk neutrality, the interest rate is always increased by ambiguity if ambiguity aversion is increasing with $U$. In the limit case with risk neutrality and constant absolute ambiguity aversion, ambiguity has no effect on the equilibrium interest rate.

However, we now show that non-increasing ambiguity aversion (DAAA) is not enough to guarantee that ambiguity reduces the socially efficient discount rate. In our counterexample, $c_0$ equals 2, and there are only $n = 2$ plausible distribution functions $F_1$ and $F_2$ for future consumption $\tilde{c}_t$. The corresponding conditional distributions are depicted in Figure ?? We assume that the these two distributions are equally likely to be the true one, i.e., $q_1 = q_2 = 1/2$. We assume that the agent has a constant relative risk aversion $\gamma = 2$, i.e., $u(c) = -c^{-1}$. We assume that the rate of pure preference for the present $\delta$ equals zero. It is easy to check that the interest rate equals $9.24\%$ in that economy if the representative agent would be neutral to ambiguity. Suppose alternatively that she has constant absolute ambiguity aversion (CAAA) with $A = 2.11$, i.e., $\phi(U) = -\exp(-2.11U)$. Then, tedious computations lead to the conclusion that the socially efficient discount rate
should be exactly zero in that economy: \( r_t = 0! \) Thus, this example demonstrates the fact that DAAA is not enough to guarantee that ambiguity about future consumption reduces the discount rate.

**FIGURE 1 MISSING**

The two equally possible distributions of future consumption in the counter example.

5 **Sufficient conditions**

In this section, we provide some sufficient conditions to guarantee that ambiguity reduces the equilibrium interest rate when the representative agent is risk-averse and has non-increasing absolute ambiguity aversion. In case the representative agent is neutral to ambiguity, the discount rate should equal

\[
\delta - \frac{1}{t} \ln \left[ \frac{Eu'(\tilde{c}_t)}{u'(c_0)} \right],
\]

where \( \tilde{c}_t \) describes future consumption, which is distributed as \((\tilde{c}_{t1}, q_1; \ldots; \tilde{c}_{tn}, q_n)\).

Under ambiguity aversion, the pricing formula (3) can be rewritten as

\[
r_t = \delta - \frac{1}{t} \ln \left[ \frac{a Eu'(\tilde{c}_t)}{u'(c_0)} \right],
\]

with

\[
a = \sum_{\theta=1}^{n} q_\theta \phi'(Eu(\tilde{c}_t)) \frac{a Eu'(\tilde{c}_t)}{\phi'(V_t(0))}
\]

and where \( \tilde{c}_t^\phi \) is a distorted probability distribution \((\tilde{c}_{t1}, q_1^\phi; \ldots; \tilde{c}_{tn}, q_n^\phi)\) of future consumption, with

\[
q_\theta^\phi = \frac{q_\theta \phi'(Eu(\tilde{c}_t))}{\sum_{\tau=1}^{n} q_\tau \phi'(Eu(\tilde{c}_t))},
\]

for \( \theta = 1, \ldots, n \). Thus ambiguity aversion reduces the discount rate if

\[
a Eu'(\tilde{c}_t^\phi) \geq Eu(\tilde{c}_t).
\]

We see from this analysis that ambiguity aversion has two effects on the discount rate. First, it has a wealth effect. To see this, define \( b \) such that
aEu'(\tilde{c}_t^\circ) = Eu'(\tilde{c}_t^\circ + b), with the property that b is positive/negative when a is smaller/larger than unity, where a is defined by equation (8). If ambiguity aversion implies that a is larger than unity, this would have a negative wealth effect (b < 0). As is well-known, a reduction in expected future wealth reduces the interest rate. From the definitions (1) and (8) of respectively \( V_t(0) \) and a, it is immediate that a is indeed larger than unity if \( \phi \) exhibits decreasing absolute ambiguity aversion. Notice that a equals unity when \( \phi \) exhibit constant absolute ambiguity aversion, so that the wealth effect disappears in that case.

In addition to the wealth effect, there is a pessimistic effect. In the pricing formula (7), the expected marginal utility is computed by using the distorted random variable \( \tilde{c}_t^\circ \) rather than the original \( \tilde{c}_t \) to describe the uncertainty over future consumption. The distortion of these implicit beliefs depends upon the degree of ambiguity aversion and is governed by rule (9). Characterizing the effect of this distortion of beliefs on the discount rate is the aim of the remainder of this section. Of course, it would be nice to have that \( \tilde{c}_t^\circ \) is dominated by \( \tilde{c}_t \) in the sense of first-degree stochastic dominance (FSD), because it would directly imply that \( Eu'(\tilde{c}_t^\circ) \) be larger than \( Eu'(\tilde{c}_t) \), since \( u' \) is decreasing. This pessimistic effect would then reduce the discount rate.

To sum up what we have at this stage, the introduction of ambiguity aversion has a wealth effect and an pessimistic effect. Under DAAA, the wealth effect is equivalent to reducing future consumption, which tends to reduce the socially efficient discount rate. But ambiguity aversion has also the effect to distort the implicit distribution of future consumption. If this distortion is pessimistic in the sense of deteriorating the implicit distribution in the sense of FSD, this pessimistic effect would also tend to reduce the socially efficient discount rate.

Because of its importance for the remainder, we will restate the following definition.

**Definition 1** Two vectors \( (x_1, ..., x_n) \) and \( (y_1, ..., y_n) \) are said to be **anticomonotonic** if, for all \( (i, j) \in \{1, n\}^2 \), \( x_i \leq x_j \) implies \( y_i \geq y_j \).

---

8\( \tilde{x} \) dominates \( \tilde{y} \) in the sense of FSD if \( Eh(\tilde{x}) \) is larger than \( Eh(\tilde{y}) \) for all functions \( h \) that are non-decreasing. Second-degree stochastic dominance is a weaker condition because it restricts functions \( h \) to be increasing and concave. The Rothschild-Stiglitz’s increases in risk correspond to the set of concave functions \( h \).
In words, anti-comonotonicity simply demands that two vectors can be ordered such that an increase along the index goes along with a lower-valued entry in the one vector and a higher-valued entry in the other. In the current framework, anti-comonotonicity will play an important role to determine which properties of the functions $u$ and $\phi$ combined with particular distributions of $\tilde{c}_{t\theta}$ yield unambiguous comparative static results.

For instance, it follows from the analysis above, that $\tilde{q}^{\phi} / q_{\theta}$ is dominated by $q$ in the sense of the monotone likelihood ratio order (MLR) if $\tilde{q}_{\theta}^{\phi} / q_{\theta}$ and $E u(\tilde{c}_{t\theta})$ are anti-comonotonic.

Observe from (9), that $\tilde{q}_{\theta}^{\phi} / q_{\theta}$ is proportional to $\phi'(E u(\tilde{c}_{t\theta}))$. Hence we can immediately state the following.

**Remark 1** Suppose without loss of generality that $E u(\tilde{c}_{t1}) \leq ... \leq E u(\tilde{c}_{tn})$.

The following two conditions are equivalent:

1. Beliefs $\tilde{q}^{\phi}$ is dominated by $q$ in the sense of the monotone likelihood ratio order, for any set of marginals $(\tilde{c}_{t1}, ..., \tilde{c}_{tn})$ satisfying the above-mentioned ranking.

2. $\phi$ is concave.

This result has a very intuitive interpretation. Ambiguity aversion is characterized by the MLR-dominated shift in the prior beliefs. In other words, it biases beliefs by favoring the worse marginals in a very specific sense: if the agent prefers marginal $\tilde{c}_{t\theta}$ than marginal $\tilde{c}_{t\theta}$, then, the ambiguity-averse representative agent increases the implicit prior probability $q_{\theta}^{\phi}$ relatively more than the implicit prior probability $q_{\theta}^{\phi}$. This gives some flesh to our terminology in which we refer to a pessimistic effect for the distortion of implicit beliefs.

At this stage, we found two requirements on $\phi$ which guarantee a negative wealth effect and a MLR-deterioration of beliefs. However, the next Lemma shows that in general this is not enough to ensure a decrease in the interest rate.

**Lemma 2** Suppose that $\phi$ exhibits non increasing absolute ambiguity aversion, $(-\phi''(U)/\phi'(U))' \leq 0$ for all $U$. Then, ambiguity reduces the discount rate if $(E u(\tilde{c}_{t\theta}))_{\theta=1,...,n}$ and $(E u'(\tilde{c}_{t\theta}))_{\theta=1,...,n}$ are anti-comonotonic.
Proof: Because $\phi'$ is decreasing, we have that $(\phi'(Eu(\tilde{c}_t)))_{\theta=1,...,n}$ and $(Eu'(\tilde{c}_t))_{\theta=1,...,n}$ are comonotonic. By the covariance rule, it implies that

$$\sum_{\theta=1}^{n} q_{\theta} \phi'(Eu(\tilde{c}_t)) \frac{Eu'(\tilde{c}_t)}{\phi'(V_t(0))} \geq a \left[ \sum_{\theta=1}^{n} q_{\theta} Eu'(\tilde{c}_t) \right].$$

By Lemma 1, we know that $a = \sum_{\theta=1}^{n} q_{\theta} \phi'(Eu(\tilde{c}_t)) / \phi'(V_t(0))$ is larger than unity under DAAA. This implies that the left-hand side of the above equality is larger than $\sum q_{\theta} Eu'(\tilde{c}_t) = Eu'(\tilde{c}_t)$. This implies that the interest rate is reduced by ambiguity aversion. ■

In other words, decreasing absolute ambiguity aversion would always reduce the interest rate if, for any pair $(\theta, \theta') \in \{1,\ldots,n\}^2$,

$$Eu(\tilde{c}_t) \leq Eu(\tilde{c}_t') \Rightarrow -Eu'(\tilde{c}_t) \leq -Eu'(\tilde{c}_t'). \quad (11)$$

Hence, we have to find joint conditions on the distribution consumption and the utility function $u$, to ensure a clear-cut effect the interest rate. For that matter we open a parenthesis on stochastic dominance orderings. In particular, apart from first-degree stochastic dominance (FSD) and second-degree stochastic dominance (SSD), we will make use of another, weaker ordering.

Definition 2 We say that $\tilde{x}$ is riskier than $\tilde{y}$ in the sense of Jewitt if the following condition is satisfied for all concave $u$: If agent $u$ prefers $\tilde{y}$ to $\tilde{x}$, then all agents more risk-averse than $u$ also prefer $\tilde{y}$ to $\tilde{x}$.

The following definition states under which conditions distribution function $F_{t\theta'}$ actually dominates $F_{t\theta}$ in the sense of Jewitt (1989).

Definition 3 Suppose that $F_{t\theta}$ does not dominate $F_{t\theta'}$ according to SSD. Then, for increasing and concave functions $u$, $F_{t\theta'}$ dominates $F_{t\theta}$ in the sense of Jewitt, if and only if there exists some $z$ in their support $[a,b]$, such that

$$\int_{a}^{x} (F_{t\theta'}(z) - F_{t\theta}(z))dz \geq 0 \quad \text{for all} \quad x \in [a,z], \quad (12)$$

$$\int_{a}^{z} (F_{t\theta'}(z) - F_{t\theta}(z))dz = 0 \quad (13)$$

$$\int_{a}^{z} (F_{t\theta'}(z) - F_{t\theta}(z))dz \quad \text{is nondecreasing} \quad \text{on} \quad [z,b]. \quad (14)$$

12
Definition 3 can be interpreted in a way that there exists a $z$ such that, conditional on the outcome being lower than $z$, $F_{t\theta'}$ dominates $F_{t\theta}$ in the sense of SSD, whereas conditional on the outcome being higher than $z$, $F_{t\theta'}$ is dominated by $F_{t\theta}$ in the sense of FSD. Observe, that second-degree stochastic dominance is indeed stronger than Jewitt’s ordering, since SSD is contained in Definition 3 as a special case when we pick $z = b$. Note, that unlike FSD or SSD, Jewitt’s ordering is compatible with a broader mean-variance trade-off, in the sense that $\tilde{c}_{t\theta'}$ might be preferred to $\tilde{c}_{t\theta}$ even though the latter has a higher mean.

Combining the above, with well-known properties of FSD and SSD enables us to state the following result.

**Lemma 3** $E u(\tilde{c}_{t\theta})_{\theta=1,\ldots,n}$ and $(E u'(\tilde{c}_{t\theta}))_{\theta=1,\ldots,n}$ are anti-comonotonic, if one of the following conditions hold,

1. The set of marginals $(\tilde{c}_{t1},\ldots,\tilde{c}_{tn})$ can be ranked according to FSD and $u$ is increasing and concave.

2. The set of marginals $(\tilde{c}_{t1},\ldots,\tilde{c}_{tn})$ can be ranked according to SSD and $u$ is increasing, concave and exhibits prudence.

3. The set of marginals $(\tilde{c}_{t1},\ldots,\tilde{c}_{tn})$ can be ranked according to Jewitt (1989) and $u$ is increasing and concave and exhibits DARA.

**Proof:** To prove that condition 1. implies anti-comonotonicity, note that, by definition, $\tilde{c}_{t\theta'}$ dominates $\tilde{c}_{t\theta}$ in the sense of FSD, if for all increasing $v$,

$$E v(\tilde{c}_{t\theta'}) \geq E v(\tilde{c}_{t\theta}).$$

Without loss of generality rank the marginals such that

$$\theta' \geq \theta \Rightarrow \tilde{c}_{t\theta'} \succeq_{\text{FSD}} \tilde{c}_{t\theta}.$$

The fact that for an increasing and concave function $u$, both $u$ and $-u'$ are increasing yields the result.

In order to prove that condition 2. yields anti-monotonicity, proceed analogously, exploiting that both $u$ and $-u'$ are increasing and concave. And similarly, 3. can be proved by exploiting that DARA implies that $-u'$ is a concave transformation of $u$. ■

Since, by assumption, our representative agent exhibits risk-aversion, the following set of Propositions combine the preceding results.
Proposition 2 Suppose that the representative agent is non increasing absolute ambiguity-averse (DAAA), and that the set of conditional probability distributions of future consumption can be ranked according to first-degree stochastic dominance. Then, ambiguity aversion reduces the socially efficient discount rate.

The ranking of conditional distributions according to first-degree stochastic dominance is a relatively restrictive condition that would be desirable to relax. Second-degree stochastic dominance (SSD) is weaker than FSD, and it contains Rothschild-Stiglitz’s increases in risk as a particular case. However, part 2 of Lemma 3 indicates that we have to impose the convexity of $u'$ to obtain the following Proposition.

Proposition 3 Suppose that the representative agent is non increasing absolute ambiguity-averse (DAAA), and that the set of conditional probability distributions of future consumption can be ranked according to second-degree stochastic dominance. Then, ambiguity aversion reduces (resp. increases) the socially efficient discount rate if $u'$ is convex (resp. concave).

Notice that in the counter-example described by Figure ??, the two random variables $\tilde{c}_t1$ and $\tilde{c}_t2$ cannot be ranked according to SSD. This is why we obtain that ambiguity aversion raises the interest rate in spite of the fact that $u'(c) = c^{-2}$ is convex.

Proposition 3, requires that the coefficient of absolute prudence in $u$ is positive. The next natural step is to consider the family of utility functions which exhibit decreasing absolute risk-aversion (DARA). Obviously, with $u$ concave, the latter implies prudence, since it requires $-u'$ to be more concave than $u$.

Proposition 4 Suppose that the representative agent is non increasing absolute ambiguity-averse (DAAA), and that the set of conditional probability distributions of future consumption can be ranked according to Jewitt. Then, ambiguity aversion reduces (resp. increases) the socially efficient discount rate if $u$ exhibits decreasing absolute risk-aversion (DARA).

Even the weakest ordering is only met with a mild requirement on risk-attitudes. The assumption of decreasing risk-aversion is widely accepted in the Economic literature and it is in particular compatible with the observation that more wealthy individuals tend to invest more in stocks.
6 Conclusion

As indicated by recent literature, parameter uncertainty might well be decisive in evaluating costs and benefits across a long time-range. This paper provides comparative statics results when introducing an ambiguity averse representative agent. In general, we cannot say that ambiguity aversion shifts the socially efficient discount down. However, we can identify sufficient conditions for a clear effect. Under DAAA and pairs of joint conditions on the risk-attitude and the stochastic ordering of future consumption, cash-flows should in effect be discounted at a smaller rate.

To our knowledge, there exists no empirical literature to correctly specify smooth ambiguity preferences yet. We might conjecture that non increasing absolute ambiguity aversion is plausible by the parallels to its well-established cousin in risk-theory, DARA. But this is all the more unsatisfactory in view of the fact that for long-term investments, ambiguity attitudes may indeed be more important than risk attitudes.

Clearly, this paper disregards many aspects of a dynamic economy. On the one hand, we do not explicitly model the evolution of growth. Hence, a natural extension would characterize the term structure under ambiguity aversion more generally than our analytical example. However, such a model should ideally incorporate the issue of evolution of information. This persistence of ambiguity and its repercussions on the interest rates demand some further attention.
References


Kimball, M.S., (1990), Precautionary savings in the small and in the large, Econometrica, 58, 53-73.


Appendix: Proof of Equation (4)

This proof relies on the well-known property that the Arrow-Pratt approximation is exact with an exponential function and a normally distributed random variable. For example, this property implies that

\[
Eu(\tilde{c}_{t\theta}) = (1 - \gamma)^{-1} E \exp \left[ (1 - \gamma) (\ln \tilde{c}_{t\theta}) \right]
\]

\[
= (1 - \gamma)^{-1} \exp \left[ (1 - \gamma) (\ln c_0 + \theta t + 0.5(1 - \gamma)\sigma^2 t) \right].
\]

Similarly, we have that

\[
Eu'(\tilde{c}_{t\theta}) = \exp \left[ -\gamma (\ln c_0 + \theta t - 0.5\gamma\sigma^2 t) \right].
\]

Because \(\tilde{\theta}\) is normally distributed, we obtain that

\[
\phi(V_t(0)) = E\phi(Eu(\tilde{c}_{t\theta})) = h k (1 - \gamma)^{-1 + \frac{\eta}{k}} E \exp \left[ (1 - \gamma)(1 - \frac{\eta}{k}) \times \right.
\]

\[
\left. \times (\ln c_0 + \tilde{\theta} t + 0.5(1 - \gamma)\sigma^2 t) \right]
\]

\[
= h k \left( \frac{1 - \gamma}{k} \right)^{-1 + \frac{\eta}{k}} \exp \left[ (1 - \gamma)(1 - \frac{\eta}{k}) \times \right.
\]

\[
\left. \times (\ln c_0 + \mu t + 0.5(1 - \gamma)\sigma^2 t + 0.5(1 - \gamma)(1 - \eta/k)\sigma_0^2 t^2) \right],
\]

which implies in turn that

\[
\phi'(V_t(0)) = hk \left( \frac{1 - \gamma}{k} \right)^{\frac{\eta}{k}} \exp \left[ -(1 - \gamma) \frac{\eta}{k} \times \right.
\]

\[
\left. \times (\ln c_0 + \mu t + 0.5(1 - \gamma)\sigma^2 t + 0.5(1 - \gamma)(1 - \eta/k)\sigma_0^2 t^2) \right].
\]

We also obtain that

\[
\phi'(Eu(\tilde{c}_{t\theta})) = hk \left( \frac{1 - \gamma}{k} \right)^{\frac{\eta}{k}} \exp \left[ -\frac{\eta}{k} (1 - \gamma) (\ln c_0 + \theta t + 0.5(1 - \gamma)\sigma^2 t) \right],
\]

so that

\[
E\phi'(Eu(\tilde{c}_{t\theta})) Eu'(\tilde{c}_{t\theta}) = h k \left( \frac{1 - \gamma}{k} \right)^{\frac{\eta}{k}} E \exp \left[ \left( \frac{\eta(\gamma - 1)}{k} - \gamma \right) (\ln c_0 + \tilde{\theta} t) + \right.
\]

\[
+ 0.5\sigma^2 t \left( \gamma^2 - \frac{\eta(1 - \gamma)^2}{k} \right) \right].
\]

18
It implies that

\[
E\phi'(Eu(\tilde{c}_{t\theta}))Eu'(\tilde{c}_{t\theta}) = \text{hk} \left( \frac{1 - \gamma}{k} \right)^2 \exp \left[ \left( \frac{\eta(\gamma - 1)}{k} - \gamma \right) \times \right. \\
\left. \times \left( \ln c_0 + \mu t + 0.5\sigma_0^2 t^2 \left( \frac{\eta(\gamma - 1)}{k} - \gamma \right) + \\
+ 0.5\sigma^2 t \left( \gamma^2 - \frac{\eta(1 - \gamma)^2}{k} \right) \right) \right]
\]

All this implies that

\[
\ln \frac{E\phi'(Eu(\tilde{c}_{t\theta}))Eu(\tilde{c}_{t\theta})}{u'(c_0)\phi'(V_t(0))} = -\mu \gamma t + 0.5\gamma^2 \sigma^2 t + 0.5\sigma_0^2 t^2 \left( \gamma^2 \left( 1 - \frac{\eta}{k} \right) + \frac{\eta}{k} \right).
\]

By equation (3), we immediately get equation (4).