Time Diversification:
Definitions and Some Closed-Form Solutions

Tao L. Wu

State University of New York (SUNY) Buffalo

Risk: Individual and Collective Decision Making Workshop
Paris, December 19, 2007
The debate

• Should older investors allocate a smaller proportion of their wealth in risky assets than younger investors?
  • Financial planners: Yes!
  • “Time diversification”: good returns cancel out bad returns over longer horizons.
  • Even if stocks outperform bonds over longer periods, it does not follow that the proportion of wealth held in risky assets should fall with one’s horizon.
  • CRRA and i.i.d. returns (Samuelson 69, Merton 69, 71)
    – Hold constant fraction of wealth in risky assets
    – Portfolio demand independent of horizons
Exceptions to the horizon-independence result

- Alternative preferences: DARA (Thorley 95; Gollier 02)
- Subsistence consumption (Samuelson 94)
  - Sinking fund in risk free asset decreases as one ages
- Mean reversion makes stock “less risky” the longer the time available for the prices to gravitate towards their long run values. (Samuelson 94)
- Investment in risky residential housing (Cocco 04)
  - Crowds out investment in stocks for young and poor
- Labor Income
  - PV of future wage income (bond-like) ↓ as one ages
  - Positive correlation, reduce investment in stocks
  - Combine with habit formation: Hump-shaped demand
Stochastic investment opportunities

• Merton 69, 71, seminal work on dynamic portfolio choice with stochastic variation in investment opportunities
  – Portfolio weights solve a nonlinear PDE, where there is no explicit solution in general.

• Most studies resort to numerical methods or log-linearization, or infinite horizon, with a few exceptions.
• Kim and Omberg 96: no inter-temporal consumption.
• Wachter 02 allows consumption during life time, but is restricted to CRRA.
• Liu 06 allows quadratic asset returns and solve the problem under CRRA up to the solution of an ODE.
Contributions of this paper

• We formalize two alternative notions of time diversification commonly confounded in the literature.
  – Cross-sectional time diversification
  – Time-series time diversification

• We provide conditions and examples with analytic solutions for both time-series and cross-sectional forms of time diversification.

• Solve in closed-form the hedging demand for a CARA investor with inter-temporal consumption and a finite horizon, when expected returns are mean-reverting and correlated with actual returns.
Cross-sectional time diversification

• Imagine two people who have the same wealth and face the same expected rate of return from the risky asset at the same time.

• Cross-sectional time diversification occurs if the younger person invests a larger proportion of his wealth in the risky asset than the older person does:

\[ \lambda(W_t, \alpha_t, t, T_2) > \lambda(W_t, \alpha_t, t, T_1). \quad T_2 > T_1. \]
Time-series time diversification

- Consider an investor at time $t$, who expects to live until $T$.
- He has wealth $W_t$ and faces the expected rate of return $\alpha_t$.
- He anticipates his demand for the risky asset at time $\tau$ with $t < \tau < T$.
- Time-series time diversification occurs if

$$E_t \lambda(W_\tau, \alpha_\tau, \tau, T) < \lambda(W_t, \alpha_t, t, T).$$
A general model of time diversification

• Risk-free asset with the constant rate of return \( r \)

• Risky asset

\[
\frac{dP}{P} = \alpha dt + \sigma_p dz_p.
\]

• The expected rate of return

\[
d\alpha = \mu_\alpha dt + \sigma_\alpha dz_\alpha.
\]

where

\[
dz_p dz_\alpha = \rho_{p_\alpha} dt, \rho_{p_\alpha} >= 0.
\]

• The proportion of wealth invested in the risky asset: \( \lambda(t) \).

• The consumer receives wage income of \( Y(t)dt \)
Solution to the maximization problem

- Agent maximizes expected lifetime utility

\[
E_0 \left\{ \int_0^T U[C(t), t] dt + \Omega[W(T), T] \right\}.
\]

- The demand for the risky asset satisfies

\[
\lambda(W, \alpha, t, T) = - \frac{J_w(\alpha - r)}{J_{ww} W \sigma_p^2} \frac{J_{w\alpha} \sigma_p}{J_{ww} W \sigma_p^2}.
\]
Time diversification with HARA preferences

• Now let’s specialize to HARA preference

\[ U(C) = \frac{1 - \gamma}{\gamma} \left( \frac{\beta C}{1 - \gamma} + \eta \right)^\gamma \]

• The portfolio demand is

\[ \lambda = \left[ \frac{\alpha - r}{\sigma_p^2} + \frac{H(\alpha, t, T) \sigma_p \alpha}{H(\alpha, t, T) \sigma_p^2} \right] \left\{ \frac{1}{1 - \gamma} + \frac{\eta + Y}{\beta r W} \left( 1 - e^{r(t-T)} \right) \right\}. \]

• \( (\eta + Y)(1 - e^{r(t-T)}) \) is the PV of wage income and “guaranteed consumption”
Conditions for cross-sectional time diversification

*Proposition 1*: Cross-sectional time diversification can occur under three circumstances:

1. \( Y > 0 \),
2. \( \eta > 0 \),
3. function \( H_\alpha / H \) depends on the horizon of the investor.
Conditions for time-series time diversification

• **Proposition 2**: Time-series time diversification can occur under three circumstances:

  • (1) $Y > 0$, or $\eta > 0$, or

  • (2) The function $H_\alpha / H$ is expected to decrease over time or

  • (3) $\alpha_t$ is expected to decline over time.
An example of time-series time diversification

- Expected return mean-reverts, uncorrelated with stock return

\[ d\alpha = \theta (\mu - \alpha)dt + \sigma_\alpha dz_\alpha \]

- Investor exhibits CRRA

\[ U[C(t), t] = e^{-\rho t} \frac{C_t^\gamma}{\gamma} \]

- Portfolio demand and the expected change

\[ \lambda_t = \frac{\alpha_t - r}{(1 - \gamma)\sigma_p^2} \]

\[ E_t\lambda_t - \lambda_t = \frac{(\mu - \alpha_t)(1 - e^{-\theta(\tau - t)})}{(1 - \gamma)\sigma_p^2} \]

- A necessary and sufficient condition for time-series time diversification to obtain: the current expected rate of return is greater than its long run mean.
An example of cross-sectional time diversification

- CARA preference

\[ U[C(t), t] = -\frac{e^{-\rho t - \eta C(t)}}{\eta}. \]

- The expected return follows an Ornstein-Uhlenbeck process from Merton 71, and correlates with the stock return

\[ d\alpha = \theta(\mu - \alpha)dt + \delta\left(\frac{dP}{P} - \alpha dt\right) = \theta(\mu - \alpha)dt + \delta \sigma dz. \]

long-run regressive adjustment  short-run adaptive learning
Closed-form solution for a finite-horizon CARA investor with intertemporal consumption

- Portfolio demand is

\[
\lambda W = \frac{\alpha - r}{\eta r \sigma^2}
\]

\[
- \frac{\delta}{\eta r \sigma^2} x_1 \left[ \frac{\theta(\mu - r)}{x_2} + (\alpha - r) \right]
\]

\[
+ \frac{\delta}{\eta r \sigma^2 (\delta + \theta)} \left\{ \frac{\theta(\mu - r)}{x_2} e^{-x_2 h} + \frac{\delta(\alpha - r)}{x_1} e^{-x_1 h} + \frac{\theta(\alpha - \mu)}{x_1} e^{-x_1 h} \right\}
\]

\[
x_1 = r + 2(\delta + \theta)
\]

\[
x_2 = r + \delta + \theta
\]
\(\delta > 0: \textbf{Case 1a}\)

- **Figure 1.** \(\alpha > \Gamma\). In this case \(X\) is globally decreasing and convex in \(h\), tending asymptotically to the demand in Merton 71, \(X_m\), as \(h \to \infty\).

\[
\Gamma = \frac{\theta \mu + \delta r}{\delta + \theta}
\]
\( \delta > 0: \textbf{Case 2a} \)

- **Figure 2.** \( \Gamma > \alpha > r \). \( X \) is again globally decreasing in \( h \), for positive values of \( h \). \( X \) may initially be convex or concave.
$\delta > 0$: Case 3a

- **Figure 3.** $r > \alpha$. $X$ is initially *increasing* in $h$. $X$ reaches a maximum, and then falls to $X_m$. 
δ<0: Case 1b

- Figure 4. $\alpha > \Gamma$. $X$ is globally increasing and concave in $h$ tending asymptotically to $Xm$ as $h \to \infty$. 
Δ<0: **Case 2b**

- **Figure 5.** $\Gamma > \alpha > r$. $X$ is again globally increasing in $h$, for positive values of $h$, $X$ may initially be convex or concave for positive $h$. 

![Graph showing $X$ vs $h$]
δ<0: **Case 3b**

- **Figure 6.** $r > \alpha$. $X$ initially decreases in $h$, reaches a minimum and then increases asymptotically to $X_m$ as $h \to \infty$. 

![Graph showing the behavior of $X$ as a function of $h$.](#)
Summary

• We formalize two alternative notions of time diversification commonly confounded in the literature.
  – Cross-sectional time diversification
  – Time-series time diversification

• We provide conditions and examples with analytic solutions for both forms of time diversification.

• Solve in closed-form the portfolio demand for a CARA investor with inter-temporal consumption and a finite horizon, when expected returns are mean reverting and correlated with stock returns. The correlation between the two is critical to the effect of the horizon on investor’s portfolio demand.