Representation results for law invariant, time consistent functions

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joint work with M. Kupper

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Theorem A

Let \((\varrho_t)_{t \in \mathbb{N}_0}\) be a law invariant, time consistent, dynamic convex risk measure. Then there is \(\gamma \in [0, \infty]\) such that

\[ \varrho_t(X) = \frac{1}{\gamma} \ln \mathbb{E}[\exp(-\gamma X) | \mathcal{F}_t], \]

the limiting cases \(\gamma = 0\) (resp. \(\gamma = \infty\)) being defined as

\[ \varrho_t(X) = \mathbb{E}[-X | \mathcal{F}], \quad \gamma = 0, \]

\[ \varrho_t(X) = \text{ess sup}_{Z \in \mathcal{P}_t} \mathbb{E}[Z(-X) | \mathcal{F}_t], \quad \gamma = \infty \]

where \(\mathcal{P}_t\) denotes the set of all positive functions \(Z\) with \(\mathbb{E}[Z | \mathcal{F}_t] = 1\).
Setting

Filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{N}_0}, \mathbb{P})\), which we assume to be of the form \(\Omega = [0, 1]^\mathbb{N}, \mathbb{P} = \lambda \otimes \mathbb{N}\), and \(\mathcal{F}_t\) generated by the first \(t\) coordinates.

Remark

Freddy Delbaen has obtained similar results in a continuous time setting and for a Brownian filtration \((\mathcal{F}_t)_{t \geq 0}\).
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a) A **dynamic convex risk measure** is a family \((\varrho_t)_{t \in \mathbb{N}_0}\) of mappings

\[
\varrho_t : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})
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such that, conditionally on \(\mathcal{F}_t\), the usual properties of a convex risk measure are satisfied (convexity, monotonicity, cash invariance).
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b) It is called **time consistent** if

\[
\varrho_0(X) = \varrho_0(-\varrho_t(X)), \quad X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})
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and it has the localisation property

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1_A \varrho_t c_t(X) = \varrho_t c_t(1_A X), \quad \text{for } A \in \mathcal{F}_t.
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\mathbbm{1}_A \varrho_t c_t(X) = \varrho_t c_t(\mathbbm{1}_A X), \quad \text{for } A \in \mathcal{F}_t.
\]

c) It is called **law invariant** if
\[
\varrho_0(X) = \varrho_0(Y) \text{ if law}(X) = \text{law}(Y).
\]
Remark

Law invariance implies the Fatou property (i.e., weak-star semi-continuity) of $\varrho_0$ (c.f. Jouini, S., Touzi 2006).
The entropic risk measure is (up to the sign) a special case of the mean value insurance premium principle (Gerber, 1979)

$$c_t(X) = u^{-1}(E[u(X)|\mathcal{F}_t])$$

where $u$ is an increasing concave function $u : \mathbb{R} \to \mathbb{R}$. 
Clearly such a premium principle \((c_t)_{t \in \mathbb{N}_0}\) is **time consistent** as, for \(0 \leq s \leq t\),

\[
c_s(c_t(X)) = u^{-1}\mathbb{E}[u(u^{-1}\mathbb{E}[u(X)|\mathcal{F}_t])|\mathcal{F}_s] = c_s(X).
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It was shown by Gerber that (under regularity assumptions on \(u\)) the function \(u\) is exponential or linear iff \(c_0(X)\) is cash invariant, i.e. \(c_0(X + m) = c_0(X) + m\), for \(m \in \mathbb{R}\).
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In general, \(c_0\) is only normalized on constants, i.e.

\[
c_0(m) = m, \quad \text{for } m \in \mathbb{R},
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and \(c_t(m) = m, \quad \text{for } m \in L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})\).
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Interpretation of \(c\): certainty equivalent.
Theorem B

Let \((c_t)_{t \in \mathbb{N}_0}\) be a family of maps \(c_t : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})\) which are normalized on constants, strictly monotone, \(\|\cdot\|_\infty\)-continuous, law invariant, and time consistent. Then there is a strictly increasing, continuous function \(u : \mathbb{R} \rightarrow \mathbb{R}\) such that

\[c_t(X) = u^{-1}(E[u(X) | \mathcal{F}_t]),\]

and \(u\) is unique up to affine transformations.
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How to find, for given \((c_t)_{t \in \mathbb{N}_0}\), the function \(u\)?
Sketch of Proof

Suppose \((\varrho_t)_{t \in \mathbb{N}_0}\) (or \((c_t)_{t \in \mathbb{N}_0}\)) satisfies the above list of axioms.

Fix \(\varepsilon > 0\) and define \(\eta_{\varepsilon}(x) \in \mathbb{R}\) implicitly by

\[
\varrho_0(x) = \varrho_0(x + \eta_{\varepsilon}(x) + \varepsilon b),
\]

where \(P[b = 1] = P[b = -1] = \frac{1}{2}\).

The number \(\eta_{\varepsilon}(x)\) may be interpreted as a certainty equivalent for the bet \(\varepsilon b\).

Remark: If \(\varrho_0\) is cash invariant, the function \(\eta_{\varepsilon}(x)\) does not depend on \(x\). For simplicity we focus on this case (Theorem A).
Suppose \((\varrho_t)_{t \in \mathbb{N}_0}\) (or \((c_t)_{t \in \mathbb{N}_0}\)) satisfies the above list of axioms. Fix \(\varepsilon > 0\) and define \(\eta_\varepsilon(x) \in \mathbb{R}\) implicitly by

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Examples

For the entropic risk measure

\[ \varrho_t(X) = \frac{1}{\gamma} \ln \mathbb{E}[\exp(-\gamma X)|\mathcal{F}_t], \]

we find

\[ \eta_\varepsilon(x) = 0, \quad \gamma = 0, \]
\[ \eta_\varepsilon(x) \approx \gamma \varepsilon^2, \quad 0 < \gamma < \infty, \]
\[ \eta_\varepsilon(x) = \varepsilon, \quad \gamma = \infty. \]
Let's start again from a general risk measure \((\varrho_t)_{t \in \mathbb{N}_0}\) satisfying the assumptions of Theorem A. We may find a sequence \((\varepsilon_k)_{k=1}^{\infty}\) tending to zero and \(\hat{\gamma} \in [0, \infty]\) such that

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\lim_{k \to \infty} \frac{\eta \varepsilon}{\varepsilon^2} = \hat{\gamma}.
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Our obvious candidate for the function \(u\) such that

\[
\varrho_t(X) = u^{-1}(\mathbb{E}[u(X)|\mathcal{F}_t])
\]

of course is

\[
u(x) = -e^{-\hat{\gamma}x}, \quad \text{for } 0 < \hat{\gamma} < \infty.
\]
But how to verify that this function indeed induces \((\varrho_t)^{t \in \mathbb{N}_0}\)?
But how to verify that this function indeed induces \((\varphi_t)_{t \in \mathbb{N}_0}\)?

We know the value of \(\varphi_0(X)\) for all random variables of the form

\[
X = x + \eta \varepsilon + \varepsilon \, b,
\]

\[
X = x + \sum_{i=1}^{n} (\eta \varepsilon + \varepsilon \, b_i),
\]

\[
X = x + \sum_{i=1}^{\tau} (\eta \varepsilon + \varepsilon \, b_i),
\]

namely \(\varphi_0(X) = x \approx \frac{1}{\hat{\gamma}} \ln \mathbb{E}[\exp(-\hat{\gamma}X)]\).

Here \((b_i)_{i=1}^{\infty}\) is an i.i.d. sequence of Bernoulli variables, and \(\tau\) is a bounded stopping time.
Remark

The expression

\[ X = x + \sum_{i=1}^{\tau} (\eta_i + \epsilon \ b_i) \]

is a discrete variant of the *Skorohod* embedding problem

\[ X = x + \int_0^\tau dW_T = x + W_\tau. \]
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In the Skorohod problem "all" random variables \( X \) can be represented as above in distribution.
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In the Skorohod problem "all" random variables \( X \) can be represented as above in distribution.

In the discrete setting, sufficiently many random variables \( X \) can be represented in distribution to show that we indeed have

\[ \varrho_0(X) = \frac{1}{\hat{\gamma}} \ln \mathbb{E}[\exp(-\hat{\gamma}X)], \]

for all \( X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \).