Abstract

We present an equilibrium model of a moral-hazard economy with a very large firm and financial markets, where a stock and bonds are traded. We show that it is optimal for the principal to forbid the agent to trade the stock; that the second-best interest rate is lower than the first-best interest rate; and that the second-best equity premium can be higher or lower than the first best equity premium. We also obtain a number of striking results: the second best cost of capital for a long-term risky project can be lower than that of the first best; and the second-best market price of risk depends not only on well-known factors such as risk aversion and production-asset volatility, but also on the agent’s marginal effort productivity and marginal cost of effort. Moreover, comparative statics suggest that if the economy is sufficiently large, the higher the productivity, the lower the market price of risk, and that low interest rates can be resulted from low agent’s effort efficiency, low-profit production opportunities, and high production risk.

Keywords: equity premium, interest rate, cost of capital, moral hazard, general equilibrium, optimal contract.
1 Introduction

We present an integrated equilibrium model of product and capital markets, where production decisions are made under moral hazard. In particular, we consider a single-firm economy with financial markets. In real life, our economy can be viewed as a benchmark close to an economy with very large firms whose corporate investment decisions nontrivially affect not only the aggregate product but systematic risks of the economy those firms belong to.¹

The importance of moral hazard to the equilibrium interaction of product and capital markets should be clear. Asset values are fundamentally based on risks and profitabilities of underlying productions which are affected by both the principal’s real investment decisions and agency problems, which are in turn affected by asset prices. Thus, any attempts to understand asset prices cannot be complete without understanding underlying production decisions and potential agency problems.

In this paper, the representative principal/investor (she) starts a firm by optimally investing in a production technology, and hires an agent (he) to manage the firm. Then, the principal issues one share to the public, based on her residual claim, and continuously manages her portfolio in financial markets trading the stock and bonds. On the other hand, the agent is allowed to trade bonds, but not the stock.

In the literature, the restriction on the agent’s stock trading is frequently imposed for tractability. See, for instance, Danthine and Donaldson [2007], Gorton and He [2006], and Albuquerque and Wang [2008]. We show that it is optimal for the principal to prohibit the agent from trading the stock. The reason is that if the agent were allowed to trade the asset he manages, he could optimally undo the contract in financial markets and exert suboptimal effort, which could result in a decrease in the principal’s expected utility. See Section B in the Appendix.

Unlike most exiting general equilibrium models with moral hazard, our model yields closed-form solutions, which enable us to focus on properties of solutions beyond the existence of equilibria. In particular, we examine equilibrium effects of moral hazard on such issues as interest rates, costs of capital for risky projects, and equity premia, taking into account the principal’s decisions on real investment, contracting and portfolio management.²

We argue that the second-best equilibrium interest rate is lower than that of the first best. This result provides an additional explanation of the well-known “riskfree rate puzzle,” raised by Weil [1989] who argues that empirically observed (riskfree) interest rates are too low to justify investors’ high risk aversion. Weil conjectures that high equity premia and low interest rates may be caused by high idiosyncratic risks of consumption risks of individual investors. We show that moral hazard problems can be a cause for low interest rates.

To see this intuitively, note that given a risky real asset, the second-best cash-flow is expected to be lower than that of the first best, because of the agency cost. Thus, if the first- and second-best interest rates were to be the same, the demand for capital in the second best would be lower than that of the first best. This means higher current consumption in the second best than it

¹Real life examples may include Chinese Petroleum in China and Samsung Group in Korea. Samsung Electronics accounts for 10 percent of the entire market capitalization of the KOSPI Index (Korea Composite Stock Price Index). According to Munhwa Ilbo newspaper, the market capitalization of Samsung group reached about 17 percent of Shanghai Composite Index. See the table in the Appendix.

²See, for instance, Cochrane [2001] and references therein for determinants of interest rates and equity premia in classical economies.
does in the first best. As a result, the principal has to put up with even lower future incomes from the second-best real investment. Motivated to smooth out her intertemporal consumption plan, the principal is willing to give up some of the current consumption for future consumption by supplying extra capital to the market. With additional capital supplied, the second-best interest rate becomes lower than that of the first best. In other words, the substitution effect from current consumption to future consumption causes the second-best interest rate to be lower level than that of the first-best level.

Next, we discuss one of the most important yet misunderstood issues in corporate finance: that is, the cost of capital and real investment decisions in the presence of agency problems. It is popularly believed that the second-best cost of capital for a (long-term) risky investment is higher than that of the first best, and thus, the second-best real investment level should be lower than the first best level. However, we argue that this belief is unwarranted, as the second-best cost of capital can be lower than that of the first best.

To be specific, we show that the first-best cost of capital for a risky real asset is the riskfree rate plus the marginal market price of the total real-asset risk, whereas the second-best cost of capital is the riskfree rate plus the marginal market price of the residual-claim risk. This structural difference in the cost of capital stems from the fact that in the first best, the principal is concerned with sharing the total real-asset risk, or ‘the whole pie,’ with the agent, whereas in the second best case, she is only interested in her own share of the total risk, or the residual risk, alone. As a result, it turns out that the second-best cost of capital is lower than that of the first best, mainly because the residual risk is only a fraction of the total real-asset risk and the second-best interest rate is lower than that of the first best.

Closely related to the issue of the cost of capital are real investment decisions. If the second-best cost of capital turns out to be lower than that of the first best, then does it imply that the second-best real-investment level should be higher than that of the first best? Unlike implications from the existing literature, the correct answer is: it depends.

To see this, let us first recall that the optimal real investment decision rule in general is to equate the marginal (net) product to the marginal cost of capital. This rule is still valid even in the second-best world. In other words, in order to understand real-investment decisions, one need to look at not only the cost of capital but the marginal product. The marginal product is determined in the product market by the principal’s real investment and the agent’s effort, whereas the marginal cost of capital is determined in the capital market through competition among investors.

For the marginal product, note that given a real investment level, the second-best net production is always lower than the first best because of the agency cost which is the compensation-risk premium ascribed to the incentive compatibility condition. Thus, in spite of the low second-best cost of capital, the second-best investment level can be lower than that of the first best, if the marginal agency cost is sufficiently high. Otherwise, an overinvestment problem can arise.

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3Stulz [1995], in a keynote speech, expounds that Japanese investors find their cost of capital lower than Americans do, because Japanese have lower agency costs of managerial discretion due to differences in the organization of firms and/or investment opportunities. This kind of view can also be found even in a textbook: Romer [1996] argues that “the agency costs arising from asymmetric information raise the cost of external finance, and therefore discourage investment.”

4Note that the agency cost occurs in the product market, reducing expected future cashflows from the asset. Since the agency cost does not arise in the capital market, it is not correct to say that the cost of capital increases because of the agency cost.
i.e., the second best investment can be larger than that of the first best.\(^5\) It is well-known that in the standard principal-agent literature, underinvestment problems can arise, as the profitability of the second-best project is lower than that of the first best.\(^6\) In this paper, it is striking that if financial markets are introduced to a standard principal-agent economy without empire-building motivations, overinvestment problems can also arise, depending on the magnitude of the agency cost as compared with the difference in the cost of capital between the first- and second-best economies.

Now, we discuss effects of moral hazard on the equity premium. Unlike the above comparisons of first-best and second-best interest rates, we find that the second-best equity premium can be either higher or lower than that of the first best. This result contrasts with numerical examples provided by both Kahn [1990] and Kocherlakota [1998]. Kahn argues that moral hazard can help explain the high equity premium puzzle first raised by Mehra and Prescott [1985], whereas Kocherlakota claims that moral hazard can deepen the puzzle with too high risk-free rate and too low equity premium. However, economic interpretations/intuitions about their numerical results were not provided.

In order to understand our result intuitively, note that the equity premium on the real asset (or the market portfolio) can be expressed as a discounted dollar risk-premium on the principal’s residual-claim risk divided by the stock price. Thus, the equity premium is positively related to the dollar risk premium, and inversely related to both the interest (discount) rate and stock price. Let us suppose that the first- and second-best real-investment levels are the same. Then, the second-best dollar-risk premium is lower than the first best, because the principal’s residual claim in the second best is a smaller fraction of the real asset than it is in the first best. Thus, the dollar risk-premium factor can contribute to a decrease in the second-best equity premium. On the other hand, the low second-best interest rate contributes to an increase in the second-best equity premium. Therefore, if the second-best interest rate is sufficiently lower than those of the first best, then the second-best equity premium can be higher than that of the first best. Otherwise, the opposite can result.

Together with the riskfree rate and equity premium, the market price of risk is another fundamental capital market variable. In the first-best (classical) world, it is well-known that the market price of risk is proportional to the two factors: the risk aversion of the principal and agent, and the total real-asset risk.\(^7\) As a result, as long as the total real-asset risk remains unchanged, whatever happens in production decisions, the market price of risk is completely independent of production decisions, as it is determined only through capital market competition. This implication is consistent with the celebrated Fisher’s theorem on the separation of consumption from real investment decisions.

However, in the presence of agency problems, we show that the second-best market price of


\(^6\)A straightforward interpretation of the project selection result in Sung [1995] implies that under moral hazard, underinvestment problems can occur in order to reduce the agency cost (the compensation-risk premium) when an additional investment in a project necessarily increases the dollar risk of the project. Similar underinvestment problems can also happen under adverse selection and moral hazard. See Sung [2005].

\(^7\)Recently, although they do not consider optimal contracts, Gorton and He [2006] also show that the market price of risk can depend on the risk aversion of agents and the ownership structure, which is consistent with one of our results.
risk depends not only on the same two factors, but on the agent’s marginal effort productivity and marginal cost of effort, because of the agent’s incentive compatibility condition. It is striking that moral hazard problems occurring in product markets can fundamentally affect Arrow-Debreu security prices (or the market price of risk) that are determined in capital markets. The reason is that in the second best, the residual claim risk is affected by the sensitivity of the compensation contract which in turn depends on both the agent’s and the firm’s production efficiency. We believe this result gives a clue about how corporate decisions on productivity can directly affect the important capital market variable, i.e., the market price of risk. Namely, changes in corporate productivity can change the capital market variable, even when the the real asset risk remains unchanged. In particular, holding the real-asset risk constant, the higher the productivity, the lower the market price of risk. A caveat is that if the productivity decisions can also change the real-asset risk, then both the first- and second-best market prices of risk can change in more complicated manners.

We obtain a number of additional comparative statics with respect to various model parameters. For example, we show that in both the first- and second-best economies, the larger the current social wealth (such as GDP (gross domestic product) or GNI (gross national income)), the lower the interest rate and the higher the market price of risk. Our comparative statics further confirm somewhat intuitive results: high interest rates can result from high managerial effort efficiency, or from either profitable or safe production opportunities. These results are intuitively clear since the demand for capital increases with the managerial effort efficiency, the expected profitability of production opportunities, and decreases in the riskiness of the asset.

This paper is related to the classical theory of investment by Fisher [1930] and Hirshleifer [1958, 1970], better known as the Fisher separation theorem of consumption and investment decisions. Our model extends the Fisherian world of consumption and investment decisions by incorporating moral hazard and portfolio management problems into the principal’s production and consumption decisions in general equilibrium.

This paper is also related to the literature on general equilibrium with moral hazard. Prescott and Townsend [1984] present a general equilibrium model of moral hazard where a central planner designs contracts for all agents, and argue that a constrained competitive equilibrium can exist and implement a (constrained) Pareto-efficient allocation. There are extensions of Prescott-Townsend’s seminal work. Bisin and Gottardi [1999] introduce financial markets to a Prescott-Townsend economy. Zame [2007] generalizes Prescott-Townsend, allowing each agent to work for many firms. However, Prescott-Townsend economies do not model conflicts of interest among members of the economies that may arise in designing contracts. Citanna and Villanacci [2002] incorporate contracting problems into a general equilibrium framework for an economy with commodity markets but without financial markets, and show the existence of equilibria. Dunthine and Donaldson [2007] model a principal-agent economy in general equilibrium from a growth theoretic perspective to examine the structure of first-best contracts.8

In spite of their important contributions to the theory of general equilibrium under moral hazard, the above studies do not provide clear guidance about properties beyond the existence of their equilibria, nor do they explain how moral hazard can affect production decisions and publicly traded asset prices in financial markets. Unlike those studies, we mainly focus on

8There is a partial equilibrium model relating moral-hazard-based product markets to asset prices. Ou-Yang [2005] assumes riskfree rates and real investment levels are exogenous, and argues that the higher sensitivity of a managerial contract lower the risk premium, which however may not be related to the equity premium puzzle appearing in the literature.
implications of product and capital equilibria on interest rates, equity premia/asset prices, production investment decisions and optimal contracts.

This paper is organized as follows: Section 2 describes the economy to be analyzed. In Section 3, we state the first- and second-best problems and provide a BSDE (Backward Stochastic Differential Equation) representation of expected exponential utility. Section 4 and 5 offer general characterization of equilibrium interest rate, market price of risk and stock price for the first- and second-best worlds. In Section 6, we consider a special case for which we solve both the first- and second-best problems in closed forms. Most of our economic intuitions are built on this section. We summarize main results of the paper in Section 7. Finally, in the Appendix, we discuss implications of restricting the agent from stock trading. The Appendix also contains most proofs of the results in the paper.

2 The Economy

Consider an economy with one representative investor (the principal) and one agent consuming numeraire goods. One may visualize an economy consisting of one large firm and many small investors who are represented by the representative investor, where the large firm is managed by the agent during a continuous time period \([0, 1]\). In real life, the economy may be associated with conglomerate-driven economies like Finland, Russia, China, and Korea. See the Appendix for shares of large companies in their domestic capital markets.

The investor is endowed with initial wealth \(M^P\) and a production opportunity. The agent is endowed with initial wealth \(M^A\) and some human capital. We assume the production opportunity is so proprietary that the ownership cannot be transferred to the agent at or before time 0, although it can be traded after time 0. The economy is uncertain with the source of uncertainty given by \(B_t\), a standard Brownian motion on probability space \((\Omega, \mathbb{P}^0)\). Let \(\mathcal{F}_t\) is an augmented sigma algebra generated by \(\{B_s^t; 0 \leq s \leq t\}\).

At time zero, the principal establishes a firm based on her production opportunity to produce numeraire goods with her initial long-term irreversible investment of \(I\), and she hires the agent to manage the firm by signing a contract, or a compensation scheme \(C_1\), which is \(\mathcal{F}_1\)-measurable. The irreversibility of the principal’s initial real investment can arise, because unlike financial investment decisions, real investment decisions like building factories may not be adjusted continuously.

After time zero, the accounting value (cumulative production) process of the firm over time before compensation to the agent, denoted by \(\{D_t\}\), evolves according to the following dynamics:

\[dD_t = g(I)dB_t^0,\]

with \(D_0(I)\) being the initial value. We assume that the process \(\{D_t\}\) is public information, and that both \(g(> 0)\) and \(D_0\) are increasing and concave in \(I\).

The agent exerts effort \(\{\mu_t\}\) during the contract period \([0, 1]\) to change the probability measure of \(\{D_t\}\) from \(\mathbb{P}^0\) to \(\mathbb{P}^\mu\) such that

\[\frac{d\mathbb{P}^\mu}{d\mathbb{P}^0} = M^\mu = \exp \left\{ -\frac{1}{2} \int_0^1 \left( \frac{f(\mu_s, I)}{g(I)} \right)^2 ds + \int_0^1 \frac{f(\mu_s, I)}{g(I)} dB_s^0 \right\},\]

where \(\{\mu_t\}\) is an \(\mathcal{F}_t\)-predictable process, and \(f(\mu_s, I)\) is concave in \((\mu_s, I)\). Assume

\[E \left[ \exp \left\{ \frac{1}{2} \int_0^1 \left( \frac{f(\mu_s, I)}{g(I)} \right)^2 ds \right\} \right] < \infty.\]
Then, by the Girsanov theorem,

\[ B^\mu_t = B^0_t - \int_0^t \frac{f(\mu_s, I)}{g(I)} ds \]

is a standard Brownian motion under \( P^\mu \), and

\[ dD_t = f(\mu_t, I) dt + g(I) dB^\mu_t. \]  \( \text{(2.1)} \)

However, \( P^\mu \) is neither observable nor verifiable. For the instantaneous effort \( \mu_t dt \), the agent incurs a personal instantaneous monetary cost of \( h(\mu_t) dt \), where \( h \) is increasing and convex. His personal value of the total cumulative cost of effort during the contract period is \( \int_0^1 h(\mu_t) dt \).

The dynamics of the cumulative cash flow process (2.1) can be interpreted as an outcome process that can be affected by both the agent’s effort \( \mu_t \) and the principal’s real investment/project selection decision, \( I \). Note that both the initial real investment by the principal and subsequent production decisions by the agent are endogenized. We believe that this feature of our model is important in order to compare asset prices between a moral-hazard and classical economies, because moral hazard problems significantly influence both initial real investment and subsequent production decisions which in turn can affect not only the dollar-productivity but dollar-risk levels of involved real assets on which stock prices critically depend.

In this paper, there are capital markets where one stock and bonds are traded. The stock market emerges right after the principal and agent sign a contract, as the principal issue one share to the public based on her residual claim, \( D_1 - C_1 \). The market value of the stock depends on investors’ belief on agents’ optimal effort levels. Let their belief be \( \{\mu^*_t\} \). We shall examine general equilibrium where this investors’ belief is fulfilled. Given the belief, the capital market is complete under probability space \( (\Omega, \mathcal{P}^{\mu^*}, \mathcal{F}_t) \) with the following risk-neutral measure \( Q \) such that

\[ \frac{dQ}{dP^{\mu^*}} = Z^\mu_1 = \exp \left\{ -\frac{1}{2} \int_0^1 \phi^2_s ds - \int_0^1 \phi_s dB^\mu_s \right\}. \]

Then, \( \{\theta_t\}, \) an \( \mathcal{F}_t \)-adapted process, is called the market-price-of-risk process. Assume

\[ E \left[ \exp \left( \frac{1}{2} \int_0^T \phi^2_s ds \right) \right] < \infty. \]

Under \( Q \), again by the Girsanov theorem,

\[ B^\theta_t = B^0_t + \int_0^t \theta_s ds \]

is a standard Brownian motion. Then, equivalently, one may say that capital markets are complete under the original observable probability space \( (\Omega, P^0, \mathcal{F}_t) \) with the following risk-neutral measure \( Q \) such that

\[ \frac{dQ}{dP^0} = \frac{dQ}{dP^{\mu^*}} \frac{dP^{\mu^*}}{dP^0} = \exp \left\{ -\frac{1}{2} \int_0^1 \phi^2_s ds - \int_0^1 \phi_s dB^0_s \right\}, \]

\(^9\)In the literature, this formulation is called the weak formulation. See Schättler and Sung [1993].

\(^{10}\)See Sung [1995] for the project-selection interpretation. Of course if \( I \) is fixed, then the dynamics of the outcome becomes similar to that of Holmstrom and Milgrom [1987].

\(^{11}\)It is well-known that when markets are complete, moral hazard problems cannot arise under strong formulations of agency problems where the agent directly chooses the drift and/or volatility of the outcome. The reason is that since markets are complete, each sample path of the Brownian motion, i.e. \( \omega \in \Omega \), can be objectively verified, and so can the agent’s controls of drifts and/or volatilities. Hence agency problems become trivial. However, this trivialization can be avoided under our weak formulation where the agent chooses a probability measure, i.e. he only indirectly chooses the drift. Thus given a realized sample path, there is no way for the principal to objectively verify the drift. Therefore, nontrivial agency problems can still exist under our weak formulation.
where \( \phi_s = \theta_s - \frac{f(\mu^*_s, I)}{g(I)} \) and
\[
B_t^0 = B_t^0 + \int_0^t \left( \theta - \frac{f(\mu^*_s, I)}{g(I)} \right) ds.
\]

Let \( E, E^\mu \) and \( E^Q \) be expectation operators under probability measures \( P, P^\mu \) and \( Q \), respectively. That is, for a random variable \( \xi \),
\[
E^Q[\xi] = E^\mu[Z_1 \xi] = E[M_1 Z_1 \xi].
\]

Throughout the paper, we assume that risk-free interest (short) rate \( r_t \) at time \( t \in [0, 1] \) is deterministic function of time \( t \). Let
\[
R_t := \exp \left( \int_0^t r_s ds \right) > 1.
\]

Then, no arbitrage implies that the stock price can be computed as follows.
\[
R_t^{-1} S_t = E^Q \left[ R_t^{-1} (D_1 - C_1) | F_t \right].
\]

That is, \( R_t^{-1} S_t \) is a \( Q \)-martingale. Thus, by the martingale representation theorem, there exists a unique square integrable process \( \tilde{\sigma}_t^S \) such that
\[
dR_t^{-1} S_t = \tilde{\sigma}_t^S dB_t^0.
\]

Note that \( \tilde{\sigma}_t^S \) can be affected by the contract \( C_1 \). Let
\[
\sigma_t^S := R_t \tilde{\sigma}_t^S.
\]

Then, the above stock price dynamics can also be written as follows.
\[
dS_t = S_t r_t dt + \sigma_t^S dB_t^0.
\]

Since we cannot guarantee \( D_1 > C_1 \) in our model, in this paper, the stock price is not necessarily positive, and there is no exponential representation of the stock price.

### 2.1 Budget Constraints

Both the principal and agent consume at discrete dates 0 and 1. Let \( c_0^i \) and \( c_1^i \) denote, respectively, the initial and terminal consumptions of an individual \( i = P, A \). Right after the capital investment and their initial consumptions, (fractional) shares of the asset (the firm stock) as well as risk-free bonds are traded in the capital market. We assume that the principal can freely trade in both stock and bond markets, and that the agent can freely trade in the bond market, i.e., to borrow and lend at (riskfree) interest rates. However, the agent is prohibited from trading the stock including all derivative markets related to the stock. The reason for this prohibition is to prevent the agent from undoing his contract in capital markets. See Proposition A.2 where it is be shown that it is optimal for the principal not to allow the agent to trade the stock. Although it is one of our main results, the proposition is provided in the Appendix for ease of exposition.

Let \( W_t^i, i = P, A \) be wealth levels at time \( t \) of individual \( i \) from capital market transactions after time 0. Then, the initial wealth levels for capital market transactions after the capital investment and their initial consumptions are as follows.
\[
W_0^P = M_P - I - c_0^P
\]
\[
W_0^A = M_A - c_0^A
\]
Since he is not allowed to trade in the stock market, dynamics of the agent’s self-financed wealth process are simply given by

\[ dW_A^t = r_t W_A^t \, dt, \quad \text{or} \quad d(R_t^{-1}W_A^t) = 0. \]

On the other hand, recall that the principal starts with one share of the firm as she issues one share to herself by investing \( I \) in the firm. Let \( \tilde{\pi}_t \) be the number of shares held at time \( t \) by the principal in addition to her initial one share. The principal chooses \( \tilde{\pi}_t \) after the managerial contract is signed. (One may alternatively view the principal as a group of identical investors who are endowed with an aggregate endowment of one share and trade among themselves after the contract with the agent is signed.) Then, dynamics of the principal’s self-financed wealth processes are as follows:

\[ dW_P^t = (W_P^t - \tilde{\pi}_t S_t) r_t \, dt + \tilde{\pi}_t \sigma_t^S \, dB_t, \quad \text{(2.2)} \]

which implies

\[ dR_t^{-1}W_P^t = \pi_P^t \, dB_t, \]

where

\[ \pi_P^t := R_t^{-1} \tilde{\pi}_t \sigma_t^S. \]

**Definition 2.1** We say that both the product and capital markets are in equilibrium if and only if the optimal expected utilities of the principal and agent exist for all \( t \in [0, 1] \), and both product and financial markets clear as follows.

\[ M_P = I + c_P^0 + W_P^0 \quad \text{(2.3)} \]
\[ M_A = c_A^0 + W_A^0 \quad \text{(2.4)} \]
\[ D_1 = c_1^P + c_1^A \quad \text{(2.5)} \]
\[ c_P^1 = W_P^1 + D_1 - C_1, \quad \text{(2.6)} \]
\[ c_A^1 = W_A^1 + C_1, \quad \text{and} \]
\[ \tilde{\pi}_t^P \equiv 0, \quad \forall t \in [0, 1]. \quad \text{(2.8)} \]

Eq.’s (2.3) to (2.7) are for market clearing at time zero and one, and Eq.(2.8) is to ensure the equity market clear for all \( t \in [0, 1] \). Note that Eq.’s (2.5) to (2.7) imply

\[ 0 = W_P^1 + W_A^1, \]

where

\[ W_P^1 = R_1W_P^0 + R_1 \int_0^1 \pi_P^s \, dB_s^\theta \]
\[ = R_1W_P^0 + R_1 \int_0^1 \pi_P^s \left( \theta - \frac{f(\mu_s^*, I)}{g(I)} \right) \, dt + R_1 \int_0^1 \pi_P^s \, dB_s^\theta \]
\[ W_A^1 = R_1W_A^0. \]

Moreover, Eq.(2.8) implies that in equilibrium

\[ \pi_P^t = 0, \quad \forall t \in [0, 1]. \quad \text{(2.9)} \]

That is, in equilibrium, the principal optimally chooses \{\( \pi_P^t \)\} such that Eq.(2.9) holds.
3 Problem Statements

Thought the paper, to denote “the first best” and “the second best, we shall use $F$ and $S$ as superscripts or subscripts on various variables, particularly on $R_1$ and $\theta$, whenever clarity is desired. Otherwise, we omit them.

Given capital market variables $\{(R_1^F, \theta_1^F)\}$, the principal’s first-best problem is stated as follows.

**Problem 1 (The First Best.)** Choose $C_1$ and $(I, c_0^P, c_0^A, \{\mu_t, \pi_t\})$ to
\[
\text{max } E^\mu \left[ -\exp\left\{ -\gamma_F c_0^P \right\} - \exp\left\{ -\gamma_F (W_1^P + D_1 - C_1) \right\} \right] \\
\text{s.t. } W_1^P = M^P - c_0^P - I + \int_0^1 r_t W_t^P dt + \int_0^1 \bar{\pi}_t \sigma_t^2 dB_t^g \\
E^\mu \left[ -\exp\left\{ -\gamma_A c_0^A \right\} - \exp\left\{ -\gamma_A (W_1^A + C_1 - \int_0^1 h(\mu_t) dt) \right\} \right] \geq L.
\]

The first constraint is the self-financing condition arising from the presence of capital markets, and the second is the agent’s participation constraint. In the first best world, both the principal and agent observe the agent’s initial consumption and effort levels as well as $\{D_t\}$. Thus, the principal can dictate the agent’s initial consumption level by writing a contract $C_1$ contingent on $c_0^A$. On the other hand, the agent’s reservation utility level $L$ is determined in the labor market. In this paper, all results up to Section 6.3.4 are unaffected by $L$. In Section 6.3.4, we assume that $L$ is determined in such a way that the net present value (NPV) of the principal’s real investment is equal to zero.

Given capital market parameter variables $\{(R_1^S, \theta_1^S)\}$, the principal’s second best problem is as follows.

**Problem 2 (The Second Best.)** Choose $C_1$ and $(I, c_0^P, c_0^A, \{\mu_t, \pi_t\})$ to
\[
\text{max } E^\mu \left[ -\exp\left\{ -\gamma_F c_0^P \right\} - \exp\left\{ -\gamma_F (W_1^P + D_1 - C_1) \right\} \right] \\
\text{s.t. } W_1^P = M^P - c_0^P - I + \int_0^1 r_t W_t^P dt + \int_0^1 \bar{\pi}_t \sigma_t^2 dB_t^g \\
\mu \in \arg \max_{\hat{\mu}} E^\hat{\mu} \left[ -\exp\left\{ -\gamma_A (W_1^A + C_1 - \int_0^1 h(\hat{\mu}_t) dt) \right\} \right] \\
E^\mu \left[ -\exp\left\{ -\gamma_A c_0^A \right\} - \exp\left\{ -\gamma_A (W_1^A + C_1 - \int_0^1 h(\mu_t) dt) \right\} \right] \geq L.
\]

The main difference of the second best problem from the first best lies in the well-known incentive compatibility condition described by the second constraint. The condition requires that the agent’s effort levels be chosen by the agent himself alone. If there are more than one optimal effort levels, then it is assumed that the agent chooses one that can improve the principal’s expected utility. Both the principal and agent are allowed to observe $c_0^A$, and $\{D_t\}$. However, the principal cannot observe and verify $\{\mu_t\}$.

It is convenient to divide the time line into two stages. The first stage occurs at time 0, when the principal and agent consume $(c_0^P, c_0^A)$, and the principal decides on capital investment $I$ and compensation scheme $C_1$. The second stage is concerned with period $(0, 1]$, during which
the agent exerts continuous effort at a total private monetary cost of \( \int_0^1 h(t) dt \) and at time 1, both the principal and agent have another round of consumptions \( (c^p_1, c^a_1) \). However, since it affects the agent’s effort decisions in the second stage, \( C_1 \) has to be designed with the agent’s second-stage responses into account.

For the principal’s problems in two stages, we also divide the agent’s participation constraint into two stages as follows.

\[
E^\mu \left[ -\exp \left\{ -\gamma A \left( W^A_1 + C_1(R, \cdot) - \int_0^1 h(\mu_t) dt \right) \right\} \right] \geq -\exp \left\{ -\gamma A R \right\},
\]

\[
-\exp \left\{ -\gamma A c^A_0 \right\} - \exp \left\{ -\gamma A R \right\} \geq L,
\]

where \( R \) is an arbitrary number for the principal to decide. Given any \( R \), the principal designs \( C_1(R, \cdot) \) such that the second-stage participation constraint holds, and then she optimally chooses \( R \) subject to the first-stage participation constraint. Consequently, both the principal’s and agent’s problems can be solved in two stages backwardly: In the second stage, given the first-stage consumption and production decisions \( (c^p_0, c^a_0, I) \) and the agent’s second-stage certainty equivalent wealth \( R \), the principal designs a compensation contract \( C_1(R, \cdot) \) and then trade shares of the stock and bonds, whereas the agent, given \( c^a_0 \) and \( C_1(R, \cdot) \), chooses effort levels and trades bonds. In the first stage, the principal decides on \( (c^p_0, c^a_0, I, R) \) and thus on \( C_1 \).

### 3.1 Backward Representation of Expected Utility

Before proceeding the analysis of problems stated in the last section, we find it convenient to represent both the principal’s and agent’s second-stage utility levels in forms of backward stochastic differential equations (BSDEs). (See Williams [2006] and Cvitanić, Wan and Zhang [2007] for BSDE methods for principal-agent problems.) In this paper, the BSDE method turns out to be particularly helpful in characterizing a unique equilibrium market-price-of-risk process \{\theta_t\} for both the first- and second-best economies, as will be seen in Propositions 4.1 and 5.3.

We utilize the following lemma for the BSDE representations.

**Lemma 3.1 (Backward representation of expected utility.)** Consider a conditional expected utility function of the following form with its certainty equivalent wealth being \( V_t \), where

\[
-\exp \{-\gamma V_t\} := E^\mu \left[ -\exp \left\{ -\gamma \left( F(B^0_t) + \int_t^1 H(\mu_s, \pi_s) ds + \int_t^1 v(\mu_s, \pi_s) dB^0_s \right) \right\} \bigg| \mathcal{F}_t \right],
\]

Then, there exists a unique \( \mathcal{F}_t \)-predictable and square integrable processes \{\( Z_t \)\} such that \( V_t \) satisfies the following BSDE.

\[
V_t = F(B^0_t) + \int_t^1 \left( H(\mu_s, \pi_s) + \left( \frac{Z_s}{\gamma} + v(\mu_s, \pi_s) \right) \frac{f}{g} - \frac{\gamma}{2} \left( \frac{Z_s}{\gamma} + v(\mu_s, \pi_s) \right)^2 \right) ds - \int_t^1 \frac{Z_s}{\gamma} dB^0_s.
\]

### 4 The First Best

We start with the second stage of Problem 1. We use Lemma 3.1 to transform the principal’s second-stage problem in the form of a BSDE.
There exists a unique global equilibrium such that the first best salary function \( C_1 \) can be represented as follows:

\[
C_1 = R - R_1 \mathcal{W}_0^A - \int_0^1 \left[ \frac{Z_A}{\gamma_A} f - h(\mu_s) - \frac{1}{2\gamma_A} (Z_s^A)^2 \right] ds + \int_0^1 \frac{Z_A}{\gamma_A} dB_s^0. \tag{4.10}
\]

Moreover, there exists a unique \( \mathcal{F}_1 \)-predictable and square integrable processes \( \{ Z_t^A \} \) such that the principal’s first-best second-stage problem is equivalent to choosing \( \{ \pi_t^P, \mu_t, Z_t^A \} \) to maximize, for all \( t \in [0, 1] \),

\[
V_t^P = R_1 (\mathcal{W}_0^P + \mathcal{W}_0^A) + D_0(I) - R - \int_t^1 \frac{Z_s^P}{\gamma_P} dB_s^0 + \int_t^1 \left\{ R_1 \pi_s^P \left( \theta_s - \frac{f_s^+ + f_s^-}{g} \right) - h(\mu_s) - \frac{1}{2\gamma_A} (Z_s^A)^2 \right. \nonumber
\]

\[
+ \left. \left( \frac{Z_s^P}{\gamma_P} + g \right) \frac{f^+}{g} + \frac{Z_s^P}{\gamma_P} R_1 \pi_s^P \right) ds. \tag{4.11}
\]

The salary representation (4.10) suggests that \( \frac{Z_s^A}{\gamma_A} \) is the sensitivity process of the contract \( C_1 \) to the outcome process \( [D_s] \). Thus, in the first best, given \( R \), the principal can design \( C_1 \) by directly choosing \( \{ \mu_s \} \) and \( \{ Z_s^A \} \) to maximize her certainty equivalent process (4.11). Applying the BSDE Comparison Theorem to (4.11), the first order conditions (FOCs) for the principal’s maximization problem are as follows.

\[
h_{\mu} = \left( \frac{Z_s^P}{\gamma_P} + g + R_1 \pi_s^P \right) \frac{f_{\mu}}{g}, \tag{4.12}
\]

\[
\theta_s = \gamma_P \left( \frac{Z_s^P}{\gamma_P} - \frac{Z_s^A}{\gamma_A} + g + R_1 \pi_s^P \right), \tag{4.13}
\]

\[
Z_s^A = \theta_s. \tag{4.14}
\]

Note that for these FOCs, we have utilized an equilibrium condition \( f^* = f \). The FOCs describe how the principal chooses decision variables \( \{ \mu_s, \pi_s^P \} \) given market parameters \( \{ \theta_t \}, R_1 \). In particular, (4.10) and (4.14) suggest that the first-best sensitivity of the contract is determined by the market price of risk \( \theta_t \), regardless of the agent’s effort levels. The reason is that in the first best, only the risk sharing of the total risk of the economy matters, and thus the sharing rule depends on the total risk, but not on the effort levels.

Thus far, we have treated the market parameters as exogenous. In the next section, we examine how they are determined in equilibrium through capital market competition, or via market clearing conditions.

4.1 Equilibrium

The following Proposition reveals the equilibrium structure of the equity-market parameter process \( \{ \theta_s \} \). The proof is omitted, as it is similar to that of the second best.

**Proposition 4.1** Suppose that \( h(\mu) = (\delta/2) \mu^2 \), and \( f(\mu, I) = \mu a(I) \). Then, given \( (I, R_1, R) \), a unique equilibrium exists such that \( Z_t^P = 0 \), \( \theta = \gamma_P + \gamma_A \), \( \theta_s = \theta \) and \( \mu_s = \mu \), where

\[
\theta = \frac{\gamma_P \gamma_A}{\gamma_P + \gamma_A} g(I), \quad \text{and} \quad h_{\mu} = f_{\mu} \quad \text{i.e.,} \quad \mu = \frac{a(I)}{\delta}.
\]
Proposition 4.1 tells us that if the drift of the production function is linear and the cost of effort is quadratic, then in equilibrium, \( \theta_s \) and \( \mu_s \) are constant over time. Under this linear-quadratic assumption, the principal’s optimal BSDE (4.11) becomes quadratic. Then, the mathematical result on the existence and uniqueness of solutions to quadratic BSDEs implies that \( \theta_s \) and thus \( \mu_s \) are constant over time.

The equilibrium pair \((\theta, \mu)\) in Proposition 4.1 is based on FOCs (4.12) and (4.14), equilibrium condition \( \pi_P^s \equiv 0 \), and \( Z_P^t \equiv 0 \). The Proposition, not surprisingly, indicates that the first best optimal effort level is determined by equating the marginal cost of effort \( h \) to the marginal expected product of effort \( f \).

However, even though the agent does not trade, the market price of risk \( \theta \) is affected by the agent’s risk aversion. This implication is consistent with Gorton and He [2006, WP]. The reason is as follows. Recall from classical asset pricing theories that the market price of risk typically depends on the representative investor’s risk aversion and the volatility of the market portfolio. In our agency world, the agent shares the outcome with the principal, and thus the principal’s residual claim on the asset is affected by the sharing rule with the agent. Since the sharing rule depends on the agent’s risk aversion, so does the volatility of the residual claim. Hence, the market price of risk is affected by the agent’s risk aversion.

If the production and cost functions are not linear-quadratic, the principal optimal BSDE may not be quadratic, and thus the existence and uniqueness of solutions to the BSDE may not be guaranteed. As a result, the uniqueness of the equilibrium may not be guaranteed, even though a constant pair of \((\theta, \mu)\) can still be consistent with an equilibrium. For the rest of this section, without losing essence of economics, we just assume the linear-quadratic case and thus \( \theta_s \) is constant over time.

4.1.1 The First-Stage Problem

FOCs (4.12) to (4.14) imply the principal’s BSDE (4.11) has a solution with \( Z_P^t \equiv 0 \) as follows:

\[
V_t^P = R_1 (W_0^P + W_0^A) + D_0(I) - R + \left\{ f - h(\mu_s) - \frac{\theta^2}{2 \gamma_A} + \left( \frac{\theta}{\gamma_A} + \frac{\theta}{\gamma_P} - g \right) \theta - \frac{\theta^2}{2 \gamma_P} \right\} (1 - t).
\]

For a complete description of the equilibrium, we now turn to the principal’s first-stage problem, which can be stated as follows. Choose \((c_0^P, I, R; c_0^A, \{\mu_t\})\) to

\[
\max - \exp \left\{ -\gamma_P c_0^P \right\} - \exp \left\{ -\gamma_P V_0^P \right\},
\]

s.t. \( -\exp \left\{ -\gamma_A c_0^A \right\} - \exp \left\{ -\gamma_A R \right\} \geq L, \]

where

\[
V_0^P = R_1 (M_P + M_A - c_0^P - c_0^A) - R + G^{FB}(I; \theta),
\]

\[
G^{FB}(I; \theta) := f - h(\mu_s) - \frac{\theta^2}{2 \gamma_A} + \left( \frac{\theta}{\gamma_A} + \frac{\theta}{\gamma_P} - g \right) \theta - \frac{\theta^2}{2 \gamma_P} - R_1 I + D_0(I),
\]

and \( \mu \) satisfies \( h(\mu) = f \). Note that the principal’s production decision \( I \) affects \( G^{FB}(I) \) which consists of several parts: expected earnings from production \( D_0(I) + f \); minus dollar cost of the production capital \( R_1 I \); expected compensation for agent’s effort \( h(\mu_s) + \frac{\theta^2}{2 \gamma_A} \); dollar return
on stock portfolio, \( \frac{a}{\gamma_p} + \frac{g}{\gamma_A} - g \) \( \theta \left( = R_1 \pi^P \theta \right) \); and the principal’s aggregate risk premium on production and portfolio risks \( \frac{g_1}{\gamma_p} \).

The substitution of the constraint into the principal’s utility function yields

\[
\max - \exp \left\{ -\gamma_P (c_0^P + V_0^P) \right\} - \exp \left\{ -\gamma_P \left[ R_1 (M_P + M_A - c^P + c^A_0) + \frac{1}{\gamma_A} \ln \left( -L - e^{-\gamma_A c^A_0} \right) + G^{FB} \right] \right\}.
\]

The FOCs are as follows:

\[
\begin{align*}
R_1 &= \exp \left\{ -\gamma_P (c_0^P - V_0^P) \right\} \quad (4.15) \\
c_0^A &= -\frac{1}{\gamma_A} \ln \left( \frac{-R_1 L}{1 + R_1} \right) \quad (4.16) \\
R &= -\frac{1}{\gamma_A} \ln \left( -\frac{L}{1 + R_1} \right) \quad (4.17) \\
\theta &= g_\theta + R_1. \quad (4.18)
\end{align*}
\]

The first FOC implies that a current consumption decision is made such that the principal’s marginal rate of substitution between current and future certainty-equivalent consumption levels is equal to \( R_1 \). The second FOC also implies that \( R_1 = \exp \left\{ -\gamma_A (c_0^A - V_0^A) \right\} \). That is, at an equilibrium interest rate \( R_1 \), both the principal’s and agent’s intertemporal marginal rates of substitution are equalized. The third FOC indicates that the higher the interest rate, the higher the agent’s certainty equivalent future consumption, \( R(= V_0^A) \).

The fourth FOC suggests that the optimal long-term real investment decision \( I \) equates the marginal product \( f_I + D_0^A(I) \) to the marginal cost of capital \( g_\theta + R_1 \). This is consistent with the well-known fact that the optimal production/real investment decision is to produce until the marginal NPV of production is equal to zero. Furthermore, this FOC also tells us that the investment decision is independent of preferences of both the principal and agent, which is consistent with Fisher’s separation theorem on the independence of the consumption and investment decisions.

Furthermore, the FOC (4.18) implies that the principal’s marginal cost of capital depends on the marginal market price of the total risk \( g_\theta \), but not on the that of her residual-claim risk. The reason is as follows. Since the first-best risk sharing, given market parameters \((R_1, \theta)\), is independent of agent’s effort decisions, the marginal market price of the price-taking principal’s residual claim risk becomes equal to that of the total risk of the real asset, and consequently, the principal in effect behaves as if she is only concerned with the total risk of the real asset, when she makes her initial real-investment decision \( I \).

Given the above FOCs, the capital market parameters \((R_1, \theta)\) are determined by the market clearing conditions as follows.

**Proposition 4.2** Let \( I(R_1, \theta) \in \arg\max_I G(I) \). Then the capital market parameters \((R_1, \theta)\) in equilibrium are jointly determined by the following two equations: \( \theta = \frac{\gamma_A g_\theta}{\gamma_A + \gamma_p} g(I) \) and

\[
M_P + M_A = I(1 + R_1) - \left( \frac{1}{\gamma_p} + \frac{1}{\gamma_A} \right) \ln(R_1) + G^{FB}(I(R_1, \theta); \theta). \quad (4.19)
\]

**Proof:** By the FOCs (4.15) to (4.17), we have

\[
(1 + R_1) (c_0^P + c_0^A) = - \left( \frac{1}{\gamma_p} + \frac{1}{\gamma_A} \right) \ln(R_1) + R_1 (M_P + M_A) + G^{FB}(I(R_1, \theta); \theta).
\]
Then, the market clearing condition, \( M_P + M_A = I + c^P_0 + c^A_0 \), implies the statement.

In Section 6, we use the two equations in Proposition 4.2 to investigate properties of \((R_1, \theta)\) for a special case. Next, we examine the equilibrium stock price.

### 4.2 Equilibrium Stock Price and Equity Premium

Once equilibrium values in \( \mu(I) \), \( I(R_1, \theta) \) and \((\theta, R_1)\) are characterized, one can easily find the first-best equilibrium stock price by computing the present value of the residual claim, i.e., \( S_0 = E_t^Q[R_1^{-1}(D_1 - C_1)] \). As a result, dynamics of the first-best stock price are given as follows.

\[
\frac{dS_t}{S_t} = \left\{ r S_t + e^{-r(1-t)} \theta \left[ g - \frac{\theta}{\gamma_A} \right] \right\} dt + e^{-r(1-t)} \left[ g - \frac{\theta}{\gamma_A} \right] dB_t^n,
\]

with its initial stock price being

\[
S_0 = R_1^{-1} \left\{ D_0(I) + \left( f(\mu, I) - h(\mu_s) - \frac{1}{2} \sigma^2 \right) + \left[ \frac{1}{\gamma_A} \right] \ln \left[ \frac{-L}{1+R_1} \right] + R_1(M_A - c^A_0) \right\}.
\]

The structure of the initial stock price \( S_0 \) suggests that it consists of the following three parts: \( e^{-r} \left\{ D_0(I) + \left( f(\mu, I) - h(\mu_s) - \frac{1}{2} \sigma^2 \right) \right\} \), a surplus from production, or the present value of future product of effort net of future compensations for the agent’s effort and risk sharing; \( e^{-r} \left[ g - \frac{\theta}{\gamma_A} \right] \theta \), a market-determined risk premium on the risky dollar-return on the stock; and \( (c^A_0 - M_A) - \frac{1}{\gamma_A} e^{-r} \ln \left[ \frac{-L}{1+R_1} \right] \), an effective second-period labor-market opportunity cost to the agent. Note that the second-period agent’s certainty equivalent wealth is \( R \left( -\frac{1}{\gamma_A} \ln \left[ \frac{-L}{1+R_1} \right] \right) \), which is determined by the principal taking into account all future incomes to the agent from both the compensation \( C_1 \) and agent’s own savings \( M_A - c^A_0 \). Thus, the present value of the agent’s second-period reservation certainty equivalent to be paid by the principal is effectively reduced to \( e^{-r} R \left( M_A - c^A_0 \right) \).

On the other hand, dynamics of the stock price imply that the stock price volatility at time \( t \) is

\[
\sigma_t^S = e^{-r(1-t)} \left[ g - \frac{\theta}{\gamma_A} \right] = e^{-r(1-t)} g \frac{\gamma A}{\gamma_P + \gamma A}.
\]

Note that the volatility comes from that of the discounted principal’s residual claim on the outcome. In order to compute the equity premium in our first best economy, let us assume \( S_0 > 0 \). Then, since the drift of the (dollar) stock price is \( r S_t + e^{-r(1-t)} \theta \left[ g - \frac{\theta}{\gamma_A} \right] \), the drift of the rate of return on the stock at time 0 can be written as \( r + \nu_0 \), where

\[
\nu_0 = \frac{\theta \left[ g - \frac{\theta}{\gamma_A} \right]}{e^{rt} S_0}.
\]

Since there is only one stock in our economy, \( \nu_0 \) represents the equity premium of the economy at time 0.

Eq.(4.21) indicates that the equity premium depends on the market price of risk, interest rate, and stock price. However, these three variables interact with each other in equilibrium. Later in Section 6, we compare this first-best equity premium with that of the second best under some simplifying assumptions.
5 The Second Best

As seen in the last section, we start the analysis of Problem 2 with its second stage. Let us define $R$ to be the agent’s second-stage certainty equivalent wealth at time 0, i.e., $V_0^A = R$, and, again, use Lemma 3.1 to represent the principal’s second-stage certainty equivalent wealth process in the form of a BSDE.

**Corollary 5.2** The second-best salary function $C_1$ in Problem 2 satisfying the agent incentive compatibility condition with an agent certainty equivalent wealth level of $R$ at time 0 can be represented as follows:

$$C_1 = R - R_1 W_0^A - \int_0^1 \left[ \frac{h_\mu}{f_\mu} f - h(\mu_s) - \frac{\gamma_A}{2} \left( \frac{h_\mu}{f_\mu} g \right)^2 \right] ds + \int_0^1 \frac{h_\mu}{f_\mu} g dB_s^0. \quad (5.22)$$

Moreover, there exists a unique $\mathcal{F}_t$-predictable and square integrable processes $\{Z^P_t\}$ such that in the second stage of Problem 2, the principal chooses $(\{\pi^P_t, \mu_t\})$ to maximize, for all $t \in [0, 1]$,

$$V^P_t = R_1 (W_0^P + W_0^A) + D_0(I) - R - \int_t^1 \frac{Z^P_s}{\gamma^P} dB^0_s$$

$$+ \int_t^1 \left[ \left( \frac{Z^P_s}{\gamma^P} + g \right) \frac{f}{g} - h(\mu_s) - \frac{\gamma_A}{2} \left( \frac{h_\mu}{f_\mu} g \right)^2 + R_1 \pi^P_s \left( \theta_s - \frac{f^*}{g} + \frac{f}{g} \right) \right. \right.$$

$$\left. \left. - \frac{\gamma^P}{2} \left( \frac{Z^P_s}{\gamma^P} - \frac{h_\mu}{f_\mu} g + R_1 \pi^P_s + g \right)^2 \right] ds. \quad (5.23)$$

The structure of the second-best salary representation (5.22) is well known since Holmstrom and Milgrom’s [1987] Brownian model. Unlike the first-best case where the sensitivity of the contract $\frac{h_\mu}{f_\mu}$ is purely determined by the market price of risk $\theta_s$ independently of the agent’s optimal effort levels, the second-best sensitivity $\frac{h_\mu}{f_\mu}$ is not independent of the agent’s optimal effort levels. As can be seen shortly, this dependence affects the riskiness of the principal’s residual-claim risk and thus the market price of risk.

On the other hand, Eq.(5.23) provides a BSDE representation for the principal’s second-stage certainty equivalent process. Note that as soon as the principal decides on $C_1$, investors in the capital market infer the agent’s optimal effort choices given $C_1$, and determine the pricing kernel as explained in Section 2. Being aware of capital market responses to her choice of $C_1$, the principal treats $f^* = f$. By applying the Comparison Theorem to her BSDE (5.23) and using $f^* = f$, the FOCs for the optimality are

$$\theta_s - \gamma^P \left( \frac{Z^P_s}{\gamma^P} - \frac{h_\mu}{f_\mu} g + R_1 \pi^P_s + g \right) = 0 \quad (5.24)$$

$$- h_\mu - \gamma_A g^2 \left( \frac{h_\mu}{f_\mu} \right) \theta_\mu \left( \frac{h_\mu}{f_\mu} \right) + \left( \frac{Z^P_s}{\gamma^P} + g \right) \frac{1}{g} f_\mu + g \theta_\mu \left( \frac{h_\mu}{f_\mu} \right) = 0. \quad (5.25)$$

As seen in the first-best case, these second-best FOCs also depend on the market-price-of-risk process $\theta_s$. In the next section, we investigate equilibrium $\theta_t$ and the principal’s first-stage problem.

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12See Schättler and Sung [1993] for general cases.
5.1 Equilibrium

First, we let $I$ fixed, and characterize the structure of $\{\theta_t\}$ by using capital market clearing conditions and the principal’s FOCs (5.24) and (5.25).

**Proposition 5.3** Suppose $f(\mu, I) = \mu a(I)$ and $h(\mu) = (\delta/2)\mu^2$. Given $(I, R_1, c_A^0, c_P^0, R)$, there exists a unique equilibrium. Moreover, in this unique equilibrium, $Z_t^P \equiv 0$, and $\theta_t$ and $\mu_t$ are constant over time such that

\[
\theta = \frac{\gamma_P^3 \gamma_A \gamma g^3}{a^2 + (\gamma_A + \gamma_P) \gamma g^2}, \quad \text{and} \quad \mu = \frac{a \left(1 + \gamma_P \delta \left(\frac{2}{3}\right)^2\right)}{\delta \left(1 + (\gamma_A + \gamma_P) \delta \left(\frac{2}{3}\right)^2\right)}.
\]

Recall that the first-best market price of risk $\theta$ depends only on the risk aversion of both the principal and agent, and the riskiness of the production $g$. However, the second-best $\theta$ hinges on not only those two factors but on $\delta$ and $a$, which are parameters for the agent’s effort and production efficiency, respectively. The reason is that unlike the first-best case, the principal’s second-best residual claim risk is not independent of the productivity, because the productivity necessarily affects the agent contract sensitivity $h_{\mu}/f_{\mu}$, and thus the volatility of the principal’s residual claim $g \left(1 - h_{\mu}/f_{\mu}\right)$ and eventually the second-best equilibrium market-price of risk.

Comparing Propositions 4.1 and 5.3, we immediately have the following corollary.

**Corollary 5.3** Given $I$, (1) the first-best market price of risk is greater than that of the second best, i.e., $\theta^F > \theta^S$, and (2) if either $a$ or $1/\delta$ increases, i.e., if productivity improves, then $\theta^S$ decreases.

Given $I$, because of the incentive compatibility condition, the second best contract $C_1$ is more sensitive to the outcome $D_1$ than the first-best contract is. Thus, the dollar volatility of the second best principal’s residual claim $g \left(1 - h_{\mu}/f_{\mu}\right)$ becomes lower than that of the first best. Hence, we have $\theta_S < \theta_F$. On the other hand, given $I$, if productivity improves, the second-best contract sensitivity increases. Consequently, the principal’s residual claim risk and thus the second-best market price of risk decreases. However, as can be seen later, when $I$ is endogenized, the comparisons become complicated.

5.1.1 The First-Stage Problem

For simplicity, let us assume the linear-quadratic case. Then by Proposition 5.3, $\theta$ turns out to be constant. Thus we only focus on cases where $\theta$ is constant, in examining the principal’s decisions in equilibrium. Suppose $\theta$ is given to be an arbitrary constant, where $\theta$, at this moment, may or may not be consistent with equilibrium. Then the unique solution to the principal’s BSDE can be found by substituting (5.24) for $R_1 \pi_x$ into (5.23) and setting $Z_t^P \equiv 0$, and it turns out to be

\[
V_0^P = R_1 \left(M_P + M_A - c_0^P - c_0^A\right) + \frac{1}{\gamma_A} \ln \left(-L - e^{-\gamma_A c_0}\right) + G^{SB}(I; \theta),
\]

where

\[
G^{SB}(I; \theta) = f - h(\mu_s) - \frac{\gamma_A}{2} \left(h_{\mu}/g\right)^2 + \left(\frac{\theta}{\gamma_P} + \frac{h_{\mu}}{f_{\mu}}g - g\right) \theta - \frac{\theta^2}{2\gamma_P} - R_1 I + D_0(I).
\]
The principal’s initial long-term real-investment \( I \) affects \( G^{SB} \) which consists of several components: expected cashflow from production \( D_0(I) + f \); minus the cost of production capital \( R_I; \) expected agent compensation, \( h(\mu_s + \frac{\gamma}{\pi'} g)^2 \); dollar return on stock portfolio, \( \left( \frac{\theta}{\pi'} + \frac{h}{\pi'} g - g \right) \); and aggregate risk premium on production and portfolio risks \( \frac{\partial^2}{\pi'^2} \).

With \( V_0^P \) given as above, the principal’s first-stage problem can be written as follows.

\[
\max_{c^e, c^h, R, I} - \exp \{-\gamma_P c_0^P \} - \exp \{-\gamma_P V_0^P \}.
\]

The forms of FOCs with respect to \((c_0^P, c_0^h, R)\) are the same as (4.15) to (4.17) of the first best case. By the Envelop Theorem, the FOC with respect to \( I \) is

\[
f_I + D'_0(I) - \gamma_A \left( \frac{h}{f_\mu} g \right) h_\mu \partial_I \left( g f_\mu \right) = R_I + \theta \left( g_I - h_\mu \partial_I \left( g f_\mu \right) \right).
\]

(5.26)

The left hand side (LHS) is the marginal product net of the marginal compensation-risk premium, and the right hand side (RHS) is the second best cost of capital consisting of one plus riskfree rate and the marginal market price of residual-claim risk. Given \( I \), the LHS is determined in the product market by the agent’s effort, and the RHS in the capital market by investors including the representative principal managing their portfolios of the real asset investment and financial assets.

Comparing the functional forms of FOCs (5.26) and (4.18), one can see that the agency problem reduces the marginal product by the marginal agent’s compensation-risk premium, and the cost of capital by the marginal market-price of the compensation risk. The marginal reduction of the second-best cost of capital occurs as the marginal compensation risk affects/reduces the residual-claim risk. Recall that the market price of the first-best residual claim risk is \( \left( g - \frac{\theta}{\gamma_A} \right) \theta \), whereas that of the second best is \( g \left( 1 - \frac{h}{\pi'} \right) \theta \). Thus, for the first-best real investment decision, the price-taking principal is concerned with the marginal market price of the production-asset risk \( g_I \theta \), because, given \( \theta \), the first-best sensitivity of the contract is unaffected by a change in the level of real investment. However, in the second best, since a change in real investment level can affect the sensitivity of the contract as well as the production-asset risk, for the second-best real investment decision, the principal has to look at marginal changes in market prices of both the production-asset and compensation risks.

In order to understand consequences of the differences in the first and second best costs of capital on real investment decisions, let us rearrange the FOC (5.26) as follows.

\[
f_I + D'_0(I) = \frac{g_I \theta + R_I}{\text{cost of capital}} + \underbrace{\left( \gamma_A \frac{h}{f_\mu} g - \theta \right)}_{\text{adjusted second-best cost of capital}} h_\mu \partial_I \left( g f_\mu \right).
\]

(5.27)

The net marginal agency cost to the principal for a marginal increase in real investment is equal to a marginal change in the compensation-risk premium, \( \frac{\gamma}{\pi'} \left( \frac{h}{\pi'} g \right)^2 \), minus marginal change in the market price of residual-claim risk related to the compensation risk, \( \frac{h}{\pi'} g \theta \theta \). (Note that as the compensation volatility increases, the residual claim volatility decreases.) Note that since \( \gamma_A \frac{h}{\pi'} g - \theta > 0 \) in equilibrium, the sign of the net agency cost is the same as the sign of \( \partial_I \left( \frac{\theta}{\pi'} \right) \).
In order to obtain further insight into the second-best cost of capital, let us suppose \( f(\mu, I) = \mu I^\alpha, \) \( g(I) = \sigma_D I^\beta \) and \( h(\mu) = (\delta/2)\mu^2. \) Then (5.27) becomes

\[
\frac{f(I) + D_0(I)}{1 + (\gamma_A + \gamma_P)\delta \sigma_D^2 I^{2(\beta - \alpha)}} = (\beta - \alpha). \tag{5.28}
\]

At a first glance, it is tempting to say that if \( (R_1, \theta, I) \) were the same for both the first- and second-best economies, and \( \beta - \alpha > 0, \) then the popular belief could be supported in the sense that the second-best cost of capital would be greater than that of the first best. Unfortunately, even this simple statement cannot be warranted in general equilibrium, simply because the capital market variables \( (R_1, \theta) \) cannot be held constant between the first- and second-best economies. In Section 6, we investigate a benchmark case of \( \alpha = \beta \) in detail.

On the other hand, as already seen in the first-best case, the market clearing conditions imply the following Proposition.

**Proposition 5.4** Let \( I(R_1, \theta) \in \arg \max_I G^{SB}(I). \) Then, equilibrium capital market parameters \( (R_1, \theta) \) are jointly determined by the following two equations: \( \theta = \gamma_P \left(1 - \frac{h_\mu}{\mu} \right) g(I) \) and

\[
M_P + M_A = (1 + R_1)I - \left(\frac{1}{\gamma_P} + \frac{1}{\gamma_A} \right) \ln(R_1) + G^{SB}(I(R_1, \theta); \theta).
\]

In Proposition 5.4, the equation for \( \theta \) is from the equity market clearing condition \( \widetilde{\pi}_1^P \equiv 0 \) and the second equation is from the initial financial market clearing condition, \( W_0^P + W_0^A = 0. \) In Section 6, we use these two equations to examine properties of \( (R_1, \theta). \)

### 5.2 Equilibrium Stock Price and Equity Premium

In order to examine the equity premium in the second-best economy, it is necessary for us to understand dynamics of the stock price. Utilizing the fact that discounted stock price is the \( Q \)-expectation of future cash flows, one can easily show that the equilibrium stock price evolves over time as follows.\(^{13}\)

\[
dS_t = \left\{ r_t S_t + e^{-r(1-t)} \left(1 - \frac{h_\mu}{\mu} \right) g\theta \right\} dt + e^{-r(1-t)} \left(1 - \frac{h_\mu}{\mu} \right) gdB_t^n,
\]

with its initial stock price given by

\[
S_0 = R_1^{-1} \left\{ D_0(I) - \left(1 - \frac{h_\mu}{\mu} \right) g\theta + \frac{1}{\gamma_A} \ln \left( \frac{-L}{1 + R_1} \right) + R_1 (M_A - c_0^\delta) + f(\mu, I) - h(\mu) - \frac{\gamma_A}{2} \left( \frac{h_\mu}{\mu} g \right)^2 \right\}.
\tag{5.29}
\]

Note that the initial second-best stock price shares the same structure as that of the first best. Thus interpretations of individual terms composing the initial second-best stock price are the same as those for the first best case.

\(^{13}\)That is, \( R_1^{-1} S_t = E^Q[I_0^{-1}(D_t - C_1)] \) where

\[
C_1 = R - R_1 (M_A - c_0^\delta) - \left[ \frac{h_\mu}{\mu} g\theta - h(\mu) - \frac{\gamma_A}{2} \left( \frac{h_\mu}{\mu} g \right)^2 \right] + \frac{h_\mu}{\mu} gB_t^n.
\]
On the other hand, the dollar volatility of the stock price at time $t$ is

$$\sigma_S^t = e^{-r(1-t)} \left( 1 - \frac{h_\mu}{f_\mu} \right) g = e^{-r(1-t)} \frac{\gamma_A \delta \sigma^2}{\alpha + (\gamma_A + \gamma_P) \delta^2} g.$$  

This equation implies that holding $I$ constant, the higher the agent’s risk aversion, the higher the stock price volatility. Holding $I$ constant, because of both risk-sharing and incentive motivations, the sensitivity of the contract decreases in the agent’s risk aversion, and thus the market value of the residual claim to the principal, i.e., the stock price, becomes riskier as the agent’s risk aversion increases. Furthermore, the stock price volatility depends not only on $g$ the outcome volatility, but on the ratio $g/a$ measuring the dollar volatility to generate one expected dollar. The larger the ratio, the higher the stock price volatility.

In order to compute the equity premium, note that the drift of the stock price process is composed of two parts: risk-free dollar return and excess dollar return. Since the equity premium is customarily measured in terms of rate of return, we define it as excess dollar return divided by the stock price. However, since the stock price in this paper can be zero, we simply compute the equity premium at time zero under the assumption that initial stock price is greater than zero. Then, the equity premium, denoted by $\nu_0$, is

$$\nu_0 = \frac{\left( 1 - \frac{h_\mu}{f_\mu} \right) g^\theta}{e^{\gamma \theta S_0}}. \quad (5.30)$$

This structure tells us that holding other things constant, the equity premium is positively related to the dollar-risk premium of the stock price and inversely related to the current stock price and interest rate, where the dollar-risk premium is a dollar-risk premium on the principal’s residual claim $D_1 - C_1$. However, these are only partial relationships, because one variable can affect the others: for example, a change in the interest rate affects both the initial stock price and the residual claim. We examine more detailed properties of first- and second-best equity premia in the next section.

6 A Special Case

In order to obtain further insight into our economy, we examine a special case where the firm has a set of project opportunities exhibiting decreasing returns to scale in both expected productivity and dollar volatility. In particular, we assume $h = \delta \mu^2$, $f = \mu I^{\hat{z}}$, $g = \sigma_D I^{\hat{z}}$ and $D_0(I) = A I^{\hat{z}}$. Thus, in this special case,

$$dD_t = I^{\hat{z}} (\mu dt + \sigma_D dB_t^\mu).$$

We interpret $\sigma_D$ as a risk measure of the production opportunity set: high $\sigma_D$ implies a set of high-risk production opportunities. Given $\sigma_D$, the principal chooses a project from project/capital-budgeting opportunities $\{(f(I), g(I), D_0(I); \sigma_D)): I \in \mathcal{R}_+\}$. Once the principal’s choice is made, the chosen project $(f(I), g(I), D_0(I); \sigma_D))$ completely determines the production function of the firm with a dollar volatility of $\sigma_D I^{\hat{z}}$. 

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We assume that $R_1 + \kappa_F > 0$, in order to ensure an interior solution for $I$. Then, substituting $(\mu, \theta)$ given in Proposition 4.1 into FOC (4.18), we have

$$I^*=\frac{A/2}{R_1+\kappa_F},$$

(6.31)

where

$$\kappa_F := \frac{1}{2} \left\{ \frac{\gamma_p \gamma_A}{\gamma_p + \gamma_A} - \frac{\sigma_D^2}{\delta} - \frac{1}{\delta} \right\}.$$  

(6.32)

One may view $\kappa_F$ as a marginal risk premium adjusted for the growth rate of the outcome, and $R_1 + \kappa_F$ as a growth-adjusted first-best cost of capital for the production asset. Eq.(6.31) implies that equilibrium $I$ directly depends on $(R_1, \kappa_F, A)$. Let $\tau$ be the aggregate risk tolerance, i.e., $\tau = \frac{1}{\gamma_p} + \frac{1}{\gamma_A}$. Then, $\kappa_F$ depends on $(\tau, \sigma_D, \delta)$.

Substituting Eq.(6.31) into the bond market clearing condition given in Proposition 4.2, we have

$$M = \Phi(\kappa_F, R_1^F, A, \tau) := \frac{A^2(1 + \kappa_F + 2R_1)}{4(R_1^F + \kappa_F)^2} - \tau \ln(R_1^F).$$

(6.33)

Condition (6.33) implies that the interest rate $R_1$ directly depends on $\kappa_F$, $M$, $A$ and $\tau$, i.e., $R_1(\kappa_F, M, A, \tau)$. Note that the interest rate in equilibrium can also depend on the risk tolerance. The reason is that given limited endowment/budget, the risk tolerance affects demand for capital for risky assets, which in turn affects demand for riskfree assets.

Comparative static results for a pair of important capital market variables $(R_1, \theta)$ are provided in Propositions 6.5 and 6.6, respectively.

**Proposition 6.5** Assume an interior solution to the special case. Then, comparative statics for the first-best economy are as follows.

1. The larger the aggregate wealth $M$, the lower the interest rate $R_1$, and the higher the real investment $I$.

2. The higher the initial production efficiency $A$, the higher the interest rate, and the lower the real investment $I$.

3. The riskier the production opportunities (high $\sigma_D$), the lower the interest rate. Moreover, as $\sigma_D$ increases, the real investment level decreases (increases), if and only if $R_1 I < (>) \tau$.

4. The higher the managerial effort efficiency $(1/\delta)$, the higher the interest rate. Moreover as $1/\delta$ increases, the real investment level increases (decreases), if and only if $R_1 I < (>) \tau$.

We shall see later all comparative statics results stated in Proposition 6.5 also hold for the second best case. The first claim is well known and its economic intuitions is almost obvious. As the income increases, the investor increases both current consumption and real investment. As the supply of money is increased with the income, the interest rate decreases.

Note that for the second to fourth claims, the aggregate income/endowment $M$ is held constant, but productivity parameters are changed. In the real-business-cycle (RBC) literature, the definition of productivity varies depending on models. In this paper, improvement in productivity/profitability of investment opportunities can occur as $A$ increases or as either $\sigma_D$ or $\delta$
decreases. As well noted in the literature, a productivity shock brings about both income and substitution effects on investors consumption/investment decisions.\footnote{See for example Romer [2005] for a textbook explanation of those effects.}

In order to see the second claim, note that an increase in $A$ can be roughly viewed as an upward parallel shift of the frontier of real investment opportunities in the dollar risk-return space with the dollar return scaled in the vertical axis. When there is productivity improvement due to an increase in $A$, demand for capital rises and so does the interest rate. However, since the parallel shift does not change the riskiness of each real investment opportunity, the growth-adjusted marginal risk premium $\kappa_F$ remains unaffected. This means the growth-adjusted cost of capital increases due to the increase of the interest rate, and thus the real investment level decreases. In other words, when $A$ increases, the income effect dominates the substitution effect and thus the real investment level decreases.

In the third claim, a productivity improvement means a decrease in $\sigma_D$, i.e., a reduction in the riskiness of each real investment opportunity. Note that holding $(R_1, \theta)$ constant, the marginal risk reduction reduces $g_I = \frac{\sigma_D}{2} I^{-\frac{1}{2}}$ more when $I$ is smaller. That is, the improvement can affect small real-investment projects/economies more than large ones, in terms of the marginal cost of capital which is the RHS of (4.18). Thus, upon a productivity shock of a decrease in $\sigma_D$, the principal has stronger incentive to increase her real investment level when $I$ is small than she does when $I$ is already large. In particular, if $R_1 I < \tau$, the principal invest more as $\sigma_D$ decreases. However, if $I$ is so large that $R_1 I > \tau$, an increase in $\sigma_D$ brings about less impact on the cost of capital, and relatively more adverse effect on the investor’s future income. Thus, as $\sigma_D$ increases, the representative investor invests more to smooth out her current and future expected consumption levels. In other words, if $R_1 I > \tau$, the substitution effect dominates the income effect in the investor’s portfolio management.

Alternatively, one may understand the third claim as follows. In general equilibrium, as $\sigma_D$ changes, not only the growth-adjusted risk premium but the interest rate changes. As $\sigma_D$ increases, the growth-adjusted risk premium on the risky real asset $\kappa_F$ increases. Then, holding $R_1$ constant, (6.31) implies the demand for capital initially decreases, which in turn decreases $R_1$ and makes real investment recover. If $R_1 I > (\leq) \tau$, as a response to a initial decrease in real investment demand, the interest rate $R_1$ decreases so fast (slowly) that $R_1 + \kappa_F$ decreases (increases) and the demand recovery more (less) than offsets its initial decrease. Hence, the riskier the production opportunities, the higher (lower) the real investment level.

In the fourth claim, a productivity improvement can also occur when the managerial effort efficiency improves, or when $1/\delta$ increases. Recall that the optimal effort $\mu = \frac{1}{2} I^{\frac{1}{2}}$ and $f_I = \frac{1}{2} I^{-\frac{1}{2}}$. Thus, the benefit of improved managerial effort efficiency is larger to a smaller (real-investment) project, in terms of the marginal product of real investment as in the LHS of (4.18). Note that both changes in $\frac{1}{2}$ and $\sigma_D$ influence the real investment level, by affecting, respectively, the LHS (marginal product) and RHS (marginal cost) of the same equation. Thus, in the fourth case, we also have a conclusion similar to that of the third claim: that is, if $R_1 I > \tau$, then the substitution effect dominates the income effect, and the principal invests less, as $1/\delta$ increases.

Furthermore, all the three claims, the second to the fourth, uniformly suggest that in spite of the above-mentioned different effects on equilibrium real investment levels, the productivity improvement always increases the equilibrium interest rate. When $R_1 I < \tau$, a high interest rate stemming from productivity improvement can be easier to understand, because the improvement induces high demand for capital which can result in a high interest rate. Even when $R_1 I > \tau$,
productivity improvement can result in a high interest rate, because although the real investment and thus demand for capital decreases, demand for current consumption increases so strongly that the aggregate demand for money increases to push up the equilibrium interest rate.

In sum, Proposition 6.5 indicates that productivity improvement results in high interest rate, but can lead to an increase or decrease in real investment. Thus, it can happen that one observes interest rates increasing when there is a decrease in real investment or a decrease in demand for capital. In particular, the third and fourth claims tell us that such a decrease can occur when the existing investment level (or the size of the economy) is sufficiently high. These results are consistent with those in the RBC literature.

The next Proposition describes how the market price of risk $\theta$ can be affected as model primitives $(M, A, \sigma_D, \delta)$ change.

**Proposition 6.6** Assume an interior solution to the special case. Then, comparative statics for the first-best economy are as follows.

1. The larger the aggregate wealth $M$, the higher the market price of risk.

2. The higher the initial production efficiency $A$, the lower (higher) the market price of risk if and only if $2R_1I > (<)\tau$.

3. If $R_1I > \tau$, then the riskier the production opportunities (high $\sigma_D$), the higher the market price of risk.

4. The higher the managerial effort efficiency, i.e. the higher the value in $1/\delta$, the higher (lower) the market price of risk, if and only if $R_1I < (>\tau$.

When the aggregate wealth increases, the real investment increases, increasing the overall risk of the investment, $g$. Since it is proportional to $g(I)$, the market price of risk increases, as $g$ increases. On the other hand, the productivity improvement can result in either an increase or decrease in the market price of risk. The main reason is that the productivity improvement can increase or decrease the real investment level as seen in Proposition 6.5. However, the proposition also suggests that if the economy is sufficiently large such that $R_1I > \tau$, then the higher the productivity (with low $\sigma_D$ or low $\delta$ or high $A$), the lower the market price of risk.

### 6.2 The second best

Note first that since $\alpha = \beta = \rho = \frac{1}{2}$, the net marginal agency cost in Eq. (5.27) turns out to be zero, and the principal’s decision on $I$ can be greatly simplified. Substituting $(\mu, \theta)$ in Proposition 5.3 into (5.27), we have

$$I^\frac{1}{2} = \frac{A/2}{R_1 + \kappa_S},$$

where

$$\kappa_S := \frac{1}{2} \left\{ \frac{\gamma_A (1 + \gamma_p \delta \sigma_D^2)}{1 + (\gamma_A + \gamma_p) \delta \sigma_D^2} \frac{\sigma_D^2}{\delta} - \frac{1}{\delta} \right\}.$$

We assume $R_1 + \kappa_S > 0$, in order to ensure an interior solution to the principal’s problem. As seen in the first best case, $\kappa_S$ can be viewed as the second-best marginal risk premium adjusted for the growth rate of the outcome, and $R_1 + \kappa_S$ as a growth-adjusted second-best cost of capital for the production asset.
Substituting (6.34) into the bond market clearing condition in Proposition 5.4, we have
\[ M = \Phi(\kappa_S, R_1^S, A, \tau) = \frac{A^2(1 + \kappa_S + 2R_1)}{4(R_1^S + \kappa_S)^2} - \tau \ln(R_1^S) \] (6.35)

**Proposition 6.7** Assume interior solutions to the special case. Then, comparative statics for interest rates and real investment levels of the second-best economy hold as stated in Proposition 6.5.\(^{15}\)

That is, moral hazard problems do not alter qualitative implications of the comparative statics resulted from the first-best economy.

Comparative statics for the second-best market price of risk are also similar to those of the first best as in Proposition 6.6 except for Part 4.

**Proposition 6.8** All statements in Proposition 6.6 also hold for the second-best market price of risk except that Part 4 is modified as follows:\(^{16}\)

4. If \( R_1^I > \tau \), then the higher the managerial effort efficiency, i.e. the higher the value in \( 1/\delta \), the lower the market price of risk.

The modification becomes necessary because unlike that of the first best case, the second-best market price of risk is affected by the agent cost efficiency, and thus the comparative statics with respect to \( 1/\delta \) becomes slightly more complicated. However, both the first- and second-best cases suggest that if the size of the economy \( I \) is sufficiently large such that \( R_1^I > \tau \), then high productivity means low market price of risk.

### 6.3 Comparing the first- and second-best solutions

In this section, we try to compare interest rates, real investment levels, initial stock prices and equity premia between the first- and second-best economies.

#### 6.3.1 Interest rates

Recall that the first- and second-best market clearing conditions in (6.33) and (6.35), respectively, can be written as follows. For \( i = F, S \), \( M = \Phi(\kappa_i, R_i^i, A, \tau) \). However,
\[ \kappa_S - \kappa_F = \frac{1}{2} \left\{ 1 + (\gamma_A + \gamma_P)\delta \sigma_D^2 \right\} (\gamma_P + \gamma_A) > 0, \]
and
\[ \Phi_{R_1} = -\frac{A^2(1 + R_1)}{2(1 + \kappa)} - \frac{\tau}{R_1} < 0, \]
\[ \Phi_\kappa = -\frac{A^2}{4(1 + \kappa)}(2 + \kappa + 3R_1). \]

Note that \( \kappa_S > \kappa_F \), mainly because the first-best marginal capital-growth rate is higher than the second best. Since \( R_1 + \kappa = \frac{A}{2}I^{-\frac{3}{2}} > 0 \), we have \( 2 + \kappa + 3R_1 > 2(1 + R_1) > 0, \) \( \Phi_\kappa < 0, \) and
\[ \frac{\partial R_1}{\partial \kappa} = -\frac{\Phi_\kappa}{\Phi_{R_1}} < 0. \] (6.36)

Therefore, we have the following proposition.

---

\(^{15}\)Note that \( \frac{\partial \kappa_S}{\partial \sigma_D} > 0, \) and \( \frac{\partial \kappa_F}{\partial \sigma_D} < 0. \) Thus, the proof for Proposition 6.5 is also valid for Proposition 6.7.

\(^{16}\)The proofs for all parts are similar to those of Proposition 6.6, except that \( \frac{\partial \kappa_S}{\partial (1/\delta)} < 0 \) where \( Y_S := \gamma_A\gamma_P\sigma_D^2 / (\frac{1}{\delta} + (\gamma_A + \gamma_P)\sigma_D^2). \)
Figure 1: Equilibrium interest rates with $M_P = 35$, $M_A = 0$, $\gamma_A = \gamma_P = 1$, $\sigma_D = 8.8$, and $A = 50$.

**Proposition 6.9** Assume interior solutions to the special case. The second-best interest rate, $R_S^1$, is lower than the first best, $R_F^1$.

This result provides a moral-hazard explanation of Weil’s [1989] riskfree rate puzzle. Figure 1 graphically illustrates Proposition 6.9, showing that the second best interest rates are lower than those of the first best. The figure is also consistent with the fourth statement of Proposition 6.5: the higher the managerial effort efficiency ($1/\delta$), the higher the interest rate.

In order to intuitively see the result of low second-best interest rates as in Proposition 6.9, note that the second-best market clearing condition (6.35) would have been identical to that of the first best if $\kappa_S = \kappa_F$. Therefore, the difference between $R_F^1$ and $R_S^1$ can be related to a difference in $\kappa$, or $\kappa_S - \kappa_F (> 0)$. When $\kappa$ is high, the value of marginal real investment is low, which in turn induces the principal to decrease the demand for investment capital. As a result, the second best equilibrium interest rate is lower than that of the first best.

Alternatively, one may attribute the low second-best interest rate to the substitution effect from the perspective of the principal’s portfolio management, as follows. Suppose the first-best risky asset and the first-best interest rate are given to the 2nd best world. Then, the principal invests less than she would in the first best. As a result, she has to consume more and expect less in the future, which may mean too large a gap between current and future consumption. Thus, highly motivated to smooth out her consumption plan, she is willing to give up current consumption for future consumption: the substitution effect. Consequently, she supplies more money to the capital market, which in turn drives the second-best interest rate down.

### 6.3.2 Costs of Capital

Proposition 6.9 enables us to compare the first- and second-best costs of capital for the long-term real investment. One may think of two approaches to comparing costs of capital: (1) compare the costs of capital for the aggregate real assets of the two economies and (2) costs of
capital for incremental real assets of the two economies.

For the first approach, one may compare \( R^P_1 + g_I(I_F)\theta_F \) with \( R^S_1 + g_I(I_S)\theta_S \). Given Proposition 6.9 for our special case, this comparison is now straightforward. Note that

\[
R^P_1 + g_I(I_F)\theta_F = R^P_1 + \frac{1}{2} \frac{\gamma_A \gamma_D}{\gamma_A + \gamma_D} \sigma_D^2
\]

\[
R^S_1 + g_I(I_S)\theta_S = R^S_1 + \frac{1}{2} \frac{\gamma_A \gamma_D \delta \sigma_D^2}{\gamma_A + \gamma_D} \sigma_D^2.
\]

Since we know \( R^P_1 > R^S_1 \) by Proposition 6.9, one can immediately see that \( R^P_1 + g_I(I_F)\theta_F > R^S_1 + g_I(I_S)\theta_S \). Thus we have the following proposition.

**Proposition 6.10** Assume interior solutions to the special case. The second-best cost of capital for the second-best real asset is lower than that of the first-best cost of capital for the first-best real asset.

For the second approach, consider a real asset with a marginal cashflow volatility of \( \gamma \). Then its costs of capital in the first- and second-best economy are \( R^P_1 + X\theta_F \) with \( R^S_1 + X\theta_S \). In this case, since \( R^P_1 > R^S_1 \), one can say the cost of capital of the risky asset is lower in the second-best economy than it is in the first-best economy if \( \theta_F > \theta_S \). Part 1 of Proposition 6.11 provides a sufficient condition for the inequality, \( \theta_F > \theta_S \).

### 6.3.3 Real Investment Levels

The inequality (6.36) can also be used to examine equilibrium investment levels. Recall that for \( \kappa = \kappa_F, \kappa_S \),

\[
I^\dagger = \frac{A}{2(R_1 + \kappa)}.
\]

Thus, one can immediately see that the second best investment level is higher than the first best, i.e., \( I_S > I_F \) if and only if \( R^S_1 + \kappa_S - R^P_1 - \kappa_F < 0 \). Recall that \( R^S_1 + \kappa_S \) is a growth-adjusted cost of capital or a growth-adjusted discount rate for the production project in the second best world. If the second-best growth-adjusted cost of capital is lower than that of the first best, then the second best investment will be higher.

What is striking is that as can be seen shortly, the second-best growth-adjusted cost of capital can actually turn out to be lower than that of the first best, and thus the second best investment level can sometimes be higher than that of the first best. In the next Proposition, we provide a sufficient condition under which the second-best cost of capital is lower than that of the first best.

**Proposition 6.11** Assume interior solutions to the special case.

1. If \( \frac{\gamma_P \gamma_A A^2}{4(\gamma_P + \gamma_A)} - 4\kappa_F < 0 \), then \( I_F > I_S \),

\[
\mu_F > \mu_S \left( 1 + \frac{\gamma_A \delta \sigma_D^2}{1 + \gamma_P \delta \sigma_D^2} \right), \quad \text{and} \quad \theta_F > \left( \frac{1}{\delta \sigma_D^2 (\gamma_A + \gamma_P)} + 1 \right) \theta_S.
\]

2. If \( \frac{\gamma_P \gamma_A A^2}{4(\gamma_P + \gamma_A)} - 4\kappa_S > 0 \) and the aggregate wealth \( M \) is given such that \( M \in Y \), then \( I_F < I_S \),

\[
\mu_F < \mu_S \left( 1 + \frac{\gamma_A \delta \sigma_D^2}{1 + \gamma_P \delta \sigma_D^2} \right), \quad \text{and} \quad \theta_F < \left( \frac{1}{\delta \sigma_D^2 (\gamma_A + \gamma_P)} + 1 \right) \theta_S.
\]
where
\[
\Upsilon := \left\{ M : \max_{\kappa \in \Lambda} \Phi(\bar{R}_1^1(\kappa), \kappa) < M < \min_{\kappa \in \Lambda} \Phi(\bar{R}_1^2(\kappa), \kappa) \right\},
\]
(6.37)

\(\bar{R}_1^1\) and \(\bar{R}_1^2\) are roots of the quadratic equation \(G(R_1(\kappa), \kappa) = 0\) such that \(\bar{R}_1^1 \leq \bar{R}_1^2\), and
\[
G(R_1, \kappa) := R_2^1 + \left(2\kappa - \frac{\gamma_P \gamma A A^2}{4(\gamma_A + \gamma_P)}\right) R_1 + \kappa^2.
\]

For an intuitive understanding of the case \(I_F < I_S\) appearing in Part 2 of Proposition 6.11, recall that the second-best growth-adjusted risk premium \(\kappa\) on risky investment is higher than that of the first best. The high growth-adjusted risk premium decreases the demand for investment capital and thus the interest rate. However as the interest rate decreases, the demand for capital increases. Thus, if the interest rate decreases faster (slower) than \(\kappa\) increases, or if the interest rate differential between the first and the second best is larger than the risk premium differential, then the second best production investment level becomes higher (lower) than that of the first best. In other words, if the interest rate differential between the first and the second best is larger (smaller) than the risk premium differential, then in the second best, the well-known substitution effect dominates (is dominated by) the income effect on the principal’s portfolio decisions more than it does (is) in the first best, the second-best real investment turns out to be higher than that of the first best.

Both Figures 2 and 3 numerically illustrate cases for \(I_F < I_S\), in which the interest rate differential between the first and the second best is larger than the real-investment risk premium differential. Figure 4 demonstrates, in one graph, a case where both \(I_F < I_S\) and \(I_F > I_S\) can occur. In Figure 4, the second-best real investment becomes higher (lower) than the first best, because the interest rate differential between the first and the second best is smaller (larger) than the real-investment risk premium differential when the risk of production opportunities is not too high (low).

### 6.3.4 Stock Prices and Equity Premia

The stock price is the present value or the discounted Q-expectation of future cash flow, \(D_1 - C_1\).

In equilibrium, the principal would be willing to invest \(I\) only when \(S_0 \geq I\), or when the NPV of the investment is nonnegative.

For our linear-quadratic case, we examine equilibrium stock prices in the first- and second-best economies where the NPV of the principal’s investment is equal to zero. That is, in equilibrium, \(S_0 = I\). Given the interest rate, the agent’s reservation utility level can affect the NPV of the principal’s investment. Our zero NPV condition implies that the agent’s reservation utility is determined in such a way to make the principal’s NPV equal to 0.\(^{17}\)

In the literature, Kahn [1990] argues that the second-best equity premium is higher than the first best, whereas Kocherlakota [1998] claims the opposite. In our linear-quadratic case, the comparison depends relative sizes of the first- and second-best equilibrium interest rates as in the following proposition.

---

\(^{17}\)Another potentially interesting way to endogenize the agent reservation utility is to make it determined in such a way that the principal becomes indifferent between selling her real investment opportunity to the agent and hiring the agent to manage the firm. We leave this possibility for future research.
Figure 2: Equilibrium equity premia and real investment levels with $M_P = 784$, $M_A = 0$, $\gamma_A = 3$, $\gamma_P = 1$, $\delta = 1$, and $A = 50$.

Figure 3: Equilibrium equity premia and real investment levels with $M_P = 1500$, $M_A = 0$, $\gamma_A = \gamma_P = 1$, $\delta = 5$, and $A = 50$. 
Proposition 6.12 (i) The second best equity premium is higher than the first best if and only if
\[
\frac{R^S}{R^F} \leq \frac{(\gamma_A + \gamma_P)^2}{\frac{1}{\delta\sigma_D^2} + (\gamma_A + \gamma_P)^2}.
\]  
(6.38)

(ii) In both first- and second-best cases, interest rates are negatively correlated with equity premia.

In order to obtain a general insight into relative sizes of the first and second best equity premia, recall from (4.21) and (5.30) that the equity premium is the discounted dollar risk premium of the principal’s residual claim divided by the stock price, and that the residual claim depends on the sharing rule between the principal and agent. Thus, there are three major components affecting the equity premium: interest rate, outcome sharing rule and stock price. However, even these three factors, in equilibrium, interact with each other in a highly complicated manner. As a result, the second best equity premium can sometimes be higher or lower than that of the first best.

Let us first suppose both the first and second best economies share the same \(I\), and thus the same \(S_0\). Then, it can be shown that the second-best dollar risk premium on the residual claim is always lower than the first best, i.e., \((g - \frac{\delta}{\gamma_A} g) \theta S < (g - \frac{\delta}{\gamma_A} \theta F)\). Thus, with \(I\) held constant, (4.21) and (5.30) imply that the second-best equity premium can be larger (smaller) than the first-best equity premium if \(R^S\) is (not) sufficiently smaller than \(R^F\). Condition (6.38) quantifies how sufficiently small the second best interest rate has to be in order for the second-best equity premium to be larger than that of the first best.

In Figure 2, the second-best equity premium is higher than the first best. In this case, the
second-best interest rate is far lower than the first best, and thus the second-best discounted dollar risk premium turns out to be higher than the first-best premium. In Figure 3, the second-best equity premium is lower than that of the first best, as the second-best interest rate is not sufficiently lower than the first best.

Figure 4 provides a numerical case where the second-best real investment level (thus the second-best stock price) can be higher than that of the first best until $\sigma_D$ reaches a threshold. However, beyond the threshold, the second best real investment level is lower than that of the first best. When $\sigma_D$ becomes higher than the threshold, although the second-best interest rate is much lower than that of the first best, the second-best equity premium is so high that the real investment becomes less attractive in the second best than it is in the first best. That is, in our equilibrium agency model, both over- and underinvestment can occur.

7 Conclusion

We have presented an integrated equilibrium model of real investment, production and portfolio management with a closed form solution. The model enables us to examine effects of moral hazard on capital/financial markets.

We have shown that moral hazard problems can result in low cost of capital for risky real assets. The main reason is that given each real investment level, the principal’s residual-claim risk is a just fraction of the total production risk, and both the second-best interest rate and market price of risk are lower than those of the first best. This result of low second-best cost of capital is in contrast with a popular belief that agency costs increase the cost of capital.

Moreover, we show that even when the real investment is endogenized in full equilibrium, the second best interest rate is lower than that of the first best. We believe this result serves as an explanation for Weil’s [1989] low (riskfree) interest rate puzzle. However, Mehra-Prescott’s [1985] high equity premium puzzle may or may not be explainable using moral hazard problems, because we find that the second best equity premium can sometimes be higher or lower than that of the first best. We have provided sufficient conditions for high second-best equity premia. In other words, under the sufficient conditions, moral hazard problems can be used to simultaneously explain both Weil’s and Mehra-Prescott’s puzzles.

Thanks to our closed form solutions, we have provided a number of striking comparative statics results, most of which can be empirically testable. In particular, the higher the current social wealth, the lower the interest rate and the higher the market price of risk. High social wealth means high money supply, thus decreasing the interest rate. However, high social wealth also means high real investment, increasing the total risk of production and thus the market price of risk.

Interest rates can also be positively related to the agent’s effort efficiency and inversely related to the riskiness of project/production opportunities. Intuitively, given monetary endowment, the interest rate is determined by the demand for capital, and the high effort efficiency increases and the riskiness decreases the demand for capital for real investment.

Our comparative statics also provide insight into some conceptual issues. We have shown that in equilibrium, the second best production investment and stock prices can sometimes be higher than those of the first best. Roughly, the growth-adjusted cost of capital (which is the sum of the interest rate and growth-adjusted risk premium) can be sometimes lower in the second best than it is in the first best, as moral hazard problems decrease the interest rate more
than they increase the growth-adjusted risk premium on the risky real asset. The second-best stock price can be higher than that of the first best for various reasons, among which a major reason is that the first-best interest rate is sometimes too much higher than the second best.

In this paper, we have assumed CARA preferences for both the principal and agent, and an outcome process driven by an arithmetic Brownian motion. As a result, the stock price can sometimes be negative. It would be interesting to have our model recast with a general class of preferences and outcome processes which produce only positive stock prices. On the other hand, our model is intended to serve as a benchmark for conglomerates-driven economies like Far Eastern countries like China, Japan and Korea. The other extreme is a moral-hazard economy like the Unites States, consisting of many firms where no individual firms may significantly affect systematic risks of the whole economy. We leave these topics for future research.
Appendix

A Table: Market Shares of Large Firms in Conglomerates-driven Economies

<table>
<thead>
<tr>
<th>Country</th>
<th>Mkt Size</th>
<th>Largest Companies</th>
<th>Mkt Value</th>
<th>Mkt Share in %</th>
</tr>
</thead>
<tbody>
<tr>
<td>China</td>
<td>$4.20</td>
<td>PetroChina</td>
<td>$0.72</td>
<td>17%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>China Mobile</td>
<td>$0.35</td>
<td>8%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Sinopec</td>
<td>$0.25</td>
<td>6%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>China Life Insurance</td>
<td>$0.20</td>
<td>5%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>China Construction Bank</td>
<td>$0.20</td>
<td>5%</td>
</tr>
<tr>
<td>Finland</td>
<td>$0.35</td>
<td>Nokia</td>
<td>$0.15</td>
<td>44%</td>
</tr>
<tr>
<td>Japan</td>
<td>$4.42</td>
<td>Toyota Motor</td>
<td>$0.20</td>
<td>4%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Mitsubishi UFJ Financial</td>
<td>$0.10</td>
<td>2%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Nintendo</td>
<td>$0.08</td>
<td>2%</td>
</tr>
<tr>
<td>Korea (South)</td>
<td>$1.03</td>
<td>Samsung Electronics</td>
<td>$0.10</td>
<td>10%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Posco</td>
<td>$0.05</td>
<td>5%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Hyundai Heavy Industries</td>
<td>$0.04</td>
<td>3%</td>
</tr>
<tr>
<td>Russia</td>
<td>$1.26</td>
<td>Gazprom</td>
<td>$0.33</td>
<td>26%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Rosneft</td>
<td>$0.10</td>
<td>8%</td>
</tr>
<tr>
<td>United Kingdom</td>
<td>$3.67</td>
<td>Royal Dutch Shell</td>
<td>$0.27</td>
<td>7%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>BP</td>
<td>$0.23</td>
<td>6%</td>
</tr>
</tbody>
</table>


B Incentives When the Agent Is Allowed to Trade the Stock

A folklore in the profession is that if the agent is allowed to trade the stock of his own firm, he can undo his incentive scheme to receive a constant salary, and consequently he has no incentives to work. This kind of implicit argument has been giving a justification for partial equilibrium agency models to prohibit the agent from trading the stock of his own firm and to focus only on contracting problems in product markets. Nevertheless, it is not completely clear how allowing the agent to trade can affect the principal’s equilibrium expected utility. Also

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18 An alternative justification can be based on the following argument. Since the agent’s trading is observable, the contract can be contingent on the agent’s transaction history in such a way that the agent’s incentives remains unchanged. Thus, allowing the agent to trade does not affect the principal’s utility. This kind of reasoning, however, may not be reasonable for continuous-time models, because contracts contingent on the agent’s private transactions history may not be realistically possible.
immediately unclear is whether the agent’s incentives would improve or worsen incentives as the agent tries to hedge his risky position in securities markets, because the agent hedging may make the performance measure less noisy. Moreover, it may be the case that the agent may not want to completely hedge away his compensation risks in the process of optimally managing his own financial/asset portfolio, and as a result, he may still have incentives to work. The latter case can be particularly true when the risk of the firm can affect systematic risk of the whole economy as is the case in this paper.

In this section, we provide a condition under which the folklore holds in equilibrium, and also argue that in general, the principal is better off with forcing the agent not to trade the asset which the agent manages. It shall be seen that if the agent is allowed to trade the stock privately, then his optimal effort level is determined independently of incentives provided by the contract but is not zero in general, and the principal is worse off than she is with agent being prohibited from trading the stock.

In order to examine an economy where the principal allows the agent to trade, we revise the condition of equity-market equilibrium from (2.8) to

\[ \tilde{\pi}_P^P + \tilde{\pi}_A^A \equiv 0. \]

We still assume the linear-quadratic case, where the outcome process is linear and the cost of effort is quadratic in \( \mu \). When the agent is fully allowed to trade the stock as well as bonds, the principal’s problem can be stated as follow.

**Problem 3** Choose \( C_1 \) and \( (\mu_t, \pi_t) \) to

\[
\begin{align*}
\max & \ E^\mu \left[ -\exp \left\{ -\gamma_P \left( W_P^P + D_1 - C_1 \right) \right\} \right] \\
\text{s.t.} & \ W_P^P = R_1 W_P^0 + \int_0^1 \tilde{\pi}_P^P dB_t^P \\
& (\mu, \pi_A) \in \arg\max_{\tilde{\mu}, \tilde{\pi}_A} E^{\tilde{\mu}} \left[ -\exp \left\{ -\gamma_A \left( W_A^1 + C_1 - \int_0^1 h(\tilde{\mu}_t) dt \right) \right\} \right] \\
& \text{s.t.} \ W_A^1 = R_1 W_A^0 + \int_0^1 \tilde{\pi}_A^A dB_t^A \\
& E^\mu \left[ -\exp \{ -\gamma_{A,\theta}^A \} - \exp \left\{ -\gamma_A \left( W_A^1 + C_1 - \int_0^1 h(\mu_t) dt \right) \right\} \right] \geq -L.
\end{align*}
\]

By the incentive compatibility condition, the agent’s expected utility is

\[
\begin{align*}
\max E^\mu & \left[ -\exp \left\{ -\gamma_A \left( C_1 + W_A^1 - \int_0^1 h(\mu_s) ds \right) \right\} \right] \\
= E^\mu & \left[ -\exp \left\{ -\gamma_A \left( C_1 + R_1 W_A^0 - \int_0^1 \left( h(\mu_s) - R_1 \pi_A^A \left( \theta_s - f^* g \right) \right) ds + \int_0^1 R_1 \pi_A^A dB_s^0 \right) \right\} \right].
\end{align*}
\]

Thus, by Lemma 3.1, the BSDE representation of the agent’s certainty equivalent wealth process \( \{V_t^A\} \) is as follows.

\[
V_t^A = C_1 + R_1 W_A^0 - \int_0^1 Z_s^A dB_s^0
\]

\[
+ \int_0^1 \left( R_1 \pi_A^A \left( \theta_s - \frac{f^*}{g} \right) - h(\mu_s) + (Z_s^A + R_1 \pi_A^A) \frac{f^*}{g} - \frac{\gamma_A}{2} (Z_s^A + R_1 \pi_A^A)^2 \right) ds. \quad (A.1)
\]

In this BSDE, note that \( f^* \) is a state variable to the agent whereas \( f \) is a control variable. The reason is that the agent’s effort choice \( \mu \) is not observable to investors, and \( f^* \) is not based
on the agent’s actual effort choices, but it simply represents investors’ expectations about the agent’s effort choices.

By applying the comparison theorem to the agent BSDE (A.1), we have the FOCs with respect to $\mu$ and $\pi_A$ for the agent’s maximization of his certainty equivalent as follows.

$$\frac{h_\mu}{f_\mu}g = Z_s^A + R_1\pi_s^A,$$

$$\theta_s - \frac{f^*}{g} + \frac{f}{g} = \gamma_A (Z_s^A + R_1\pi_s^A) = 0.$$

Substituting the first FOC into the second, we have

$$\theta_s - \frac{f^*}{g} + \frac{f}{g} = \gamma_A \frac{h_\mu}{f_\mu}g = 0.$$

This condition implies that when he is allowed to trade, the agent’s effort decision is determined independently of the contract incentives. When $f^* \neq f$, the market is in disequilibrium, because investors’ expectations about the agent’s effort levels are not fulfilled. If $f^* > (\leq) f$, public investors overvalue (undervalue) the stock, and the agent is motivated to decrease (increase) his share holdings, which means low (high) incentives to work.

When the market is in equilibrium, we have $f^* = f$ and the agent’s optimal effort is determined by

$$\theta_s = \gamma_A \frac{h_\mu}{f_\mu}g.$$ (A.2)

In other words, in equilibrium, the agent’s effort levels are simply determined by the market-price-of-risk process $\{\theta_s\}$, not by the contract sensitivity. It also means that given a contract with arbitrary incentives, the agent can simply undo them and construct his own incentives through his own capital market transactions. It is striking, however, that the agent does not completely undo the contract incentives, and let himself exposed to some incentives to work. The reason is that as long as $\theta$ is positive, the agent can be rewarded for taking risks an thus wants to maintain certain risk exposure for portfolio management/capital allocation purposes. As he maintains a certain amount of exposure, he will have incentives to work only to the extent of the risk exposure.

However, this implication can also be used to defend the folklore discussed in the introduction. If $\theta$ is zero, (or if compensation risks are idiosyncratic,) then the agent will not have incentives to work given any incentive contract. The reason is that exposure to idiosyncratic risks is not rewarded in capital markets, and thus he simply unloads the compensation risk through capital market transactions. As a result, he will end up with receiving a risk-free compensation and he has no incentives to work.

Thus, the above results are summarized as follows.

**Proposition A.1** When he is allowed to trade the stock and bonds, the agent’s optimal level of effort in equilibrium is determined such that $\theta_s = \gamma_A \frac{h_\mu}{f_\mu}g$, regardless of $C_1$.

**Corollary A.1** (Folklore) If the outcome risk $\{B_t^\theta\}$ is idiosyncratic such that $\theta_t \equiv 0$, then the agent’s optimal effort in equilibrium is zero, i.e., $\mu_t \equiv 0$, regardless of $C_1$.

Nevertheless, we allow $\theta$ to be greater than or equal to zero. From (A.1), the contract $C_1$
can be represented as follows:

\[-C_1 = -R + R_1W_0^A - \int_0^1 Z_s^A dB_s^0
+ \int_0^1 \left( R_1 \pi_s + \frac{f^*}{g} \right) - h(\mu_s) + (Z_s^A + R_1 \pi_s^A) \frac{f}{g} - \frac{\gamma_A}{2} \left( Z_s^A + R_1 \pi_s^A \right)^2 \right) ds.\]

Then the principal’s terminal wealth is

\[W_1^P + D_1 - C_1 = D_0 - R + R_1W_0^A - \int_0^1 \left( \frac{h_\mu}{f_\mu} g - R_1 \pi_s^A - R_1 \pi_s^P - g \right) dB_s^0
+ \int_0^1 \left( R_1 \pi_s^A + R_1 \pi_s^P \right) \left( \theta_s - \frac{f^*}{g} \right) - h(\mu_s) + \frac{h_\mu}{f_\mu} f - \frac{\gamma_A}{2} \left( \frac{h_\mu}{f_\mu} g \right)^2 \right) ds.\]

Define the principal’s certainty equivalent wealth \( V_t^P \) as follows:

\[-e^{-\gamma_P V_t^P} = E^\mu \left[ \exp \left\{ -\gamma_P \left( R_1 \left( W_0^A + W_0^P \right) + D_0 - R - \int_t^1 \left( \frac{h_\mu}{f_\mu} g - R_1 \pi_s^A - R_1 \pi_s^P - g \right) dB_s^0
+ \int_0^1 \left( R_1 \pi_s^A + R_1 \pi_s^P \right) \left( \theta_s - \frac{f^*}{g} \right) - h(\mu_s) + \frac{h_\mu}{f_\mu} f - \frac{\gamma_A}{2} \left( \frac{h_\mu}{f_\mu} g \right)^2 \right) \right\} \bigg| \mathcal{F}_t \right]. \]

Recall that the agent’s effort is uniquely determined by the agent’s incentive compatibility condition as in Proposition A.1. Thus, when the agent is allowed to trade, the principal has no room to choose effort levels for the agent. Thus, the principal only chooses \( \pi_s^P \).

By Lemma 3.1, \( V_t^P \) can be represented as follows.

\[V_t^P = R_1 \left( W_0^A + W_0^P \right) + D_0 - V_0^A - \int_t^1 Z_s^P dB_s^0
+ \int_0^1 \left( R_1 \pi_s^A + R_1 \pi_s^P \right) \left( \theta_s - \frac{f^*}{g} \right) - h(\mu_s) + \frac{h_\mu}{f_\mu} f - \frac{\gamma_A}{2} \left( \frac{h_\mu}{f_\mu} g \right)^2 \right) ds
+ \int_0^T \left\{ \left( Z_s^P - \frac{h_\mu}{f_\mu} g + R_1 \pi_s^A + R_1 \pi_s^P + g \right) \frac{f}{g} - \frac{\gamma_P}{2} \left( Z_s^P - \frac{h_\mu}{f_\mu} g + R_1 \pi_s^A + R_1 \pi_s^P + g \right)^2 \right\} ds. \]

\[(A.3)\]

Since the principal treats \( f^* = f \), the Comparison Theorem implies that the FOC is

\[\theta_s - \gamma_P \left( Z_s^P - \frac{h_\mu}{f_\mu} g + R_1 \pi_s^A + R_1 \pi_s^P + g \right) = 0.\]

Since \( \mu \) is linear in \( \theta \) and \( \pi_s^P + \pi_s^A = 0 \) in equilibrium, we have

\[\frac{\theta_s}{\gamma_P} = \left( 1 - \frac{h_\mu}{f_\mu} \right) g.\]

Therefore, the substitution of (A.2), implies that \( \theta_t \) is constant over time at \( \theta \), where

\[\theta = \frac{\gamma_A \gamma_P}{\gamma_A + \gamma_P} g.\]

That is, when the agent is allowed to trade freely, the second-best market price of risk becomes the same as that of the first best. The net second-best contract sensitivity \( Z_t^A + R_1 \pi_t^A \), which
is equal to the sum of the contract sensitivity \( Z^A_t \) and the agent share position in the stock, is equal to that of the first best risk-sharing contract, i.e., \( Z^A_t + R_1 \pi^A_t = \frac{\gamma P}{\gamma A + \gamma P} g \), for all \( t \in [0, 1] \). This result is intuitive in the sense that when the agent is allowed to freely trade, there is no way for the principal to induce her desired level of the agent’s effort, because the agent can always undo the contract, and re-balance his risky position in securities markets according to his own preference. Then the agent’s net position in the risky asset will be determined as if both the principal and agent competitively trade without restrictions. As a result, both the principal and agent’s positions in the risky asset coincide with the first-best competitive equilibrium allocations.

Since the principal’s FOC implies \( \theta \) is linear in \( Z^P_t \) in equilibrium, the principal’s BSDE is quadratic in \( Z^P_t \). Note that the principal’s quadratic BSDE has a solution with \( Z^P_t \equiv 0 \). The solution is

\[
V^P_t = R_1 \left( W^A_0 + W^P_0 \right) + D_0 - R \\
+ \left\{ f - h(\mu_s) - \frac{\gamma A}{2} \left( \frac{h_\mu}{f_\mu} g \right)^2 - \frac{\gamma P}{2} \left( 1 - \frac{h_\mu}{f_\mu} \right)^2 g^2 \right\} (1 - t) \tag{A.4}
\]

where \( \mu \) is constant over time. Therefore, by the uniqueness of solutions to quadratic BSDEs, (A.4) is the unique solution to the principal’s BSDE (A.3) in equilibrium.

Comparing the two expected utilities (A.5) and (A.4), which are, respectively, the principal’s expected utilities with and without agent trading restrictions, one can see that both utilities are given in identical forms, and that in the latter case, the principal only chooses \( \pi^P \), whereas in the former case, she can choose \( \mu \) as well as \( \pi^P \). Therefore, the principal is better off with restricting the agent from trading the stock, as stated in the following proposition.

**Proposition A.2** The principal’s expected utility with restriction on the agent’s stock trading is always greater than or equal to that without the restriction.

This proposition provides a justification for our principal to forbid the agent from his stock trading.

### C Proof of Lemma 3.1

Let

\[
J_t := -\exp \left\{ -\gamma \left( V_t + \int_0^t H(\mu_s, \pi_s) \, ds + \int_0^t v(\mu_s, \pi_s) \, dB^0_s \right) \right\}.
\]

Since \( J_t \) is a \( P^\mu \)-martingale, by the martingale representation theorem, there exists a unique square-integrable and predictable process \( \{Z_s\} \) such that

\[
J_t = -\exp \left\{ -\gamma \left( F(B^0_t) + \int_0^t H(\mu_s, \pi_s) \, ds + \int_0^t v(\mu_s, \pi_s) \, dB^0_s \right) \right\} + \int_0^t Z_s \, dB^\mu_s.
\]

Thus,

\[
dJ_t = -J_t \tilde{Z}_t \, dB^\mu_t = -J_t \tilde{Z}_t \left( dB^0_t - \frac{f}{g} \, dt \right).
\]

On the other hand, the definition of \( J_t \) implies

\[
dJ_t = -\gamma J_t \left( dV_t + H(\mu_t, \pi_t) \, dt + v(\mu_t, \pi_t) dB^0_t \right) + \frac{\gamma^2}{2} J_t \left( \langle dV_t, dB^0_t \rangle + 2 \langle dV_t, v_t dB^0_t \rangle + v^2 \, dt \right).
\]

35
Hence,
\[ dV_t + H(\mu_t, \pi_t)dt + v(\mu_t, \pi_t)dB_t^0 - \frac{\gamma}{2} \left( 2\langle dV_t, v_t dB_t^0 \rangle + v_t^2 dt \right) = \frac{\tilde{Z}_t}{\gamma} \left( dB_t^0 - \frac{f}{g} dt \right), \]
and
\[ \langle dV_t, dV_t \rangle = \left( \frac{\tilde{Z}_t}{\gamma} - v(\mu_t, \pi_t) \right)^2 dt, \quad \text{and} \quad \langle dV_t, v_t dB_t^0 \rangle = \left( \frac{\tilde{Z}_t}{\gamma} - v(\mu_t, \pi_t) \right) v_t dt. \]

That is
\[ dV_t = - \left( H(\mu_t, \pi_t) + \frac{\tilde{Z}_t}{\gamma} f - \frac{\gamma}{2} \left( \frac{\tilde{Z}_t}{\gamma} \right)^2 \right) dt + \left( \frac{\tilde{Z}_t}{\gamma} - v(\mu_t, \pi_t) \right) dB_t^0. \]

Let \( \frac{1}{\gamma} Z_t := \frac{1}{\gamma} \tilde{Z}_t - v. \)

Then, by substitution, the assertion follows.

\section*{D Proof of Corollary 4.1}

Lemma 3.1 implies that given admissible \( \mu \), the second-stage agent’s expected utility with the certainty equivalent wealth being \( R \) can be represented as follows.

\[ V_A^0 = R = R_1 W_A^0 + C_1 + \int_0^1 \left[ \frac{Z_A^s f}{\gamma A} - h(\mu_s) - \frac{1}{2\gamma A} (Z_A^s)^2 \right] ds - \int_0^1 \frac{Z_A^s}{\gamma A} dB_s^0. \]

This relationship immediately reveals the structure of the compensation function \( C_1 \) as stated in the Lemma.

Substituting the above salary representation (4.10) for \( C_1 \) into the principal’s problem, we define the principal’s certainty equivalent wealth process \( V_P^0 \) as follows:

\[ - \exp \left\{ -\gamma_P V_P^0 \right\} = E^\mu \left[ - \exp \left\{ -\gamma_P \left( W_P^0 + D_1 - C_1 \right) \right\} \mid \mathcal{F}_t \right] \]
\[ = E^\mu \left[ - \exp \left\{ -\gamma_P \left( R_1 \left( W_P^0 + W_A^0 \right) + D_0(I) - R \right. \right. \right. \]
\[ + \left. \left. \left. \int_t^1 \left[ \frac{Z_A^s f}{\gamma A} - h(\mu_s) - \frac{1}{2\gamma A} (Z_A^s)^2 + R_1 \pi_s^P \tilde{g} + \frac{f^*}{g} \right] ds \right. \right. \right. \]
\[ - \left. \int_t^1 \left( \frac{Z_A^s}{\gamma A} - R_1 \pi_s^P \right) dB_s^0 \right) \] \[ \mid \mathcal{F}_t \right]. \]

Then, given \((c_P^0, I)\), the principal’s second-stage problem is to choose \((\mu_s, \pi_s^P, Z_A^s)\) to maximize \(- \exp \left\{ -\gamma_P V_P^0 \right\} \). By Lemma 3.1, the certainty equivalent process for the principal’s future unrealized wealth can be represented as as stated in (4.11). That is, at each time \( t > 0 \), the principal maximizes her conditional certainty equivalent wealth \( V_P^0 \) by choosing \((\pi_s^P, \mu_t, Z_A^s)\).
E Proof of Corollary 5.2

The incentive compatibility condition tells us that given $C_1$, the agent chooses his effort levels \( \{\mu_s\} \) to maximize his own certainty equivalent wealth, \( V_0^A \) where

\[
V_t^A = R_1 W_0^P + C_1 + \int_t^1 \left[ \frac{Z_A^A f}{\gamma_A} - h(\mu_s) - \frac{1}{2\gamma_A} (Z_A^A)^2 \right] ds - \int_t^1 \frac{Z_A^A}{\gamma_A} dB_0^P.
\]

By the comparison theorem, the FOC for the maximization is

\[
\frac{Z_A^A}{\gamma_A} = h_\mu \frac{f}{f_\mu} g.
\]

Thus, we have the representation of the second-best salary function as stated in (5.22).

Substituting the salary representation into the principal’s second-stage certainty equivalent wealth \( V_0^P \), we have

\[
\max \left\{ \exp \left\{ -\gamma_P V_0^P \right\} \right\} = \mathbb{E}^{\mu} \left[ \exp \left\{ -\gamma_P \left( W_0^P + D_1 - C_1 \right) \right\} \right]
\]

\[
= \mathbb{E}^{\mu} \left[ \exp \left\{ -\gamma_P \left( R_1 (W_0^P + W_A^A) + D_0(I) - R ight. \right.ight.
\]

\[
+ \int_0^1 \left[ \frac{h_\mu}{f_\mu} f - h(\mu_s) - \gamma_A \left( \frac{h_\mu}{f_\mu} g \right)^2 + R_1 \pi_s^P \left( \theta - f^* \right) \right] ds
\]

\[
- \int_0^1 \left[ \frac{h_\mu}{f_\mu} g - R_1 \pi_s^P - g \right] dB_0^s \right] \right\}
\]

Thus, by Lemma 3.1, the principal’s certainty equivalent process can be represented by a BSDE as stated in (5.23).

F Proof of Proposition 5.3

Under the stated assumptions, by FOCs (5.24) and (5.25) and the equilibrium condition \( \pi_s^P \equiv 0 \), we have

\[
\mu_s = \frac{\theta}{\gamma_A} \left( \frac{a}{\delta g} \right)^2 \left\{ \frac{\delta g}{a} + \frac{1}{\gamma_P} \frac{a}{g} \right\}.
\]

This implies \( \mu_s \) is linear in \( \theta_s \). Then, the equilibrium condition \( \pi_s^P \equiv 0 \), the linearity of \( \mu_s \) and FOC (5.24) imply \( \theta_s \) is linear in \( Z_s^P \), and thus \( \mu_s \) is linear in \( Z_s^P \) as well.

On the other hand, since \( \mu_s \) and \( \theta_s \) are linear in \( Z_s^P \), the principal’s BSDE (5.23) is quadratic in \( Z_s^P \). Furthermore, if \( Z_s^P \equiv 0 \), FOCs (5.24) and (5.25) and the equilibrium condition \( \pi_s^P \equiv 0 \) imply that both \( \mu_s \) and \( \theta_s \) are constant over time such that \( \mu_t = \mu \), and \( \theta_t = \theta \) for all \( t \) where \( (\mu, \theta) \) are given as stated in the proposition. Moreover, with the pair of constants \( (\mu_s, \theta) \), BSDE (5.23) has a solution as follows.

\[
V_t^P = R_1 (W_0^P + W_A^A) + D_0(I) - R
\]

\[
+ \left[ f - h(\mu_s) - \gamma_A \left( \frac{h_\mu}{f_\mu} g \right)^2 - \frac{1}{2\gamma_P} \theta^2 \right] \left( 1 - t \right).
\]
Thus, by the uniqueness of the solution to the quadratic BSDE, this solution is unique. Therefore, in capital market equilibrium, there exists a unique equilibrium with $\mu_t = \mu$, such that $\theta_t = \theta$ for all $t$. ■

G  Proof of Proposition 6.5

Let the RHS of condition (6.33) denoted by $\Phi(\kappa, R_1, \tau, A)$. Note that

\[
\Phi_{R_1} = -\frac{A^2(1 + R_1)}{2(R_1 + \kappa)^3} - \frac{\tau}{R_1} < 0, \quad \Phi_A = \frac{A(1 + \kappa + 2R_1)}{2(R_1 + \kappa)^2} > 0
\]

\[
\Phi_\kappa = -\frac{A^2}{4(R_1 + \kappa)^3} (2 + \kappa + 3R_1) < 0.
\]

Then, the condition implies that

\[
\frac{\partial R_1}{\partial M} = \frac{1}{\Phi_{R_1}} < 0, \quad \text{and} \quad \frac{\partial R_1}{\partial A} = \frac{\Phi_A}{\Phi_{R_1}} > 0.
\]

Therefore, these inequalities together with (6.31) imply the first and second claims.

Moreover,

\[
\frac{\partial R_1}{\partial \sigma_D} = \frac{\Phi_\kappa}{\Phi_{R_1}} \frac{\partial \kappa}{\partial \sigma_D} > 0,
\]

\[
\frac{\partial R_1}{\partial (1/\delta)} = \frac{\Phi_\kappa}{\Phi_{R_1}} \frac{\partial \kappa}{\partial (1/\delta)} > 0,
\]

\[
\frac{\partial}{\partial \sigma_D} (R_1 + \kappa) = \frac{1}{\Phi_{R_1}} \left( I - \frac{\tau}{R_1} \right) \frac{\partial \kappa}{\partial \sigma_D},
\]

\[
\frac{\partial}{\partial (1/\delta)} (R_1 + \kappa) = \frac{1}{\Phi_{R_1}} \left( I - \frac{\tau}{R_1} \right) \frac{\partial \kappa}{\partial (1/\delta)}.
\]

Since $\frac{\partial \kappa}{\partial \sigma_D} > 0$, and $\frac{\partial \kappa}{\partial (1/\delta)} < 0$, and we have the third and fourth claims. ■

H  Proof of Proposition 6.6

Note that $\theta = \frac{1}{\gamma} \varphi(I) = \frac{\sigma_F}{\tau} I^\frac{2}{3}$. In order to preserve the structure of the proof so that it can be utilized for the proof of Proposition 6.8, let $Y_F := \frac{\sigma_F}{\tau}$. Then $\theta = Y_F I^\frac{2}{3}$, and we have

\[
\frac{\partial \theta}{\partial M} = Y \frac{\partial I^\frac{2}{3}}{\partial M} > 0
\]

\[
\frac{\partial \theta}{\partial A} = Y \frac{1}{2(R_1 + \kappa)} \frac{1}{\Phi_{R_1}} \left( 2I - \frac{\tau}{R_1} \right)
\]

\[
\frac{\partial \theta}{\partial \sigma_D} = \frac{A}{2(R_1 + \kappa)^2} \left\{ \frac{\partial Y}{\partial \sigma_D} (R_1 + \kappa) - \frac{Y}{\Phi_{R_1}} \left( I - \frac{\tau}{R_1} \right) \frac{\partial \kappa}{\partial \sigma_D} \right\}
\]

\[
\frac{\partial \theta}{\partial (1/\delta)} = \frac{A}{2(R_1 + \kappa)^2} \left\{ \frac{\partial Y}{\partial (1/\delta)} (R_1 + \kappa) - \frac{Y}{\Phi_{R_1}} \left( I - \frac{\tau}{R_1} \right) \frac{\partial \kappa}{\partial (1/\delta)} \right\}.
\]

Since $\Phi_{R_1} < 0$, $\frac{\partial Y}{\partial \sigma_D} > 0$, $\frac{\partial \kappa}{\partial \sigma_D} > 0$, and $\frac{\partial \kappa}{\partial (1/\delta)} < 0$, the assertions follow. ■
I Proof of Proposition 6.11

Our sufficient condition relies on the set $\Upsilon$. Thus, we first need to establish the nonemptiness of $\Upsilon$.

**Lemma A.1** Choose $a$ and $b$ such that $a < b$, $G(a, \kappa_S) < 0$, $G(b, \kappa_S) < 0$, and $\Phi(b, \kappa_F) < \Phi(a, \kappa_S)$. Suppose $M$ is given such that $\Phi(b, \kappa_F) < M < \Phi(a, \kappa_S)$. Then, $M \in \Upsilon$ and $R_1(\kappa_F), R_1(\kappa_S) \in [a, b]$.

**Remark:** Note that conditions for Lemma A.1 can be satisfied if model parameters satisfy the following inequality: $(1 + \kappa_S)^2 (4 + \kappa_F) - (3 + \kappa_S) (2 + \kappa_F)^2 < 0$.

**Proof:** Note that $\forall \kappa \in \Lambda$, $G(R_1^1(\kappa), \kappa) = G(R_1^2(\kappa), \kappa) = 0$, and $G(R_1(\kappa), \kappa) < 0$ for $R_1(\kappa) \in [R_1^2(\kappa), R_1^1(\kappa)]$. However, since $G_\kappa > 0$, $\forall \kappa \in [\kappa_F, \kappa_S]$, we have

\[
G(a, \kappa) \leq G(a, \kappa_S) < 0,
\]

\[
G(b, \kappa) \leq G(b, \kappa_S) < 0.
\]

Thus, $\forall \kappa \in \Lambda$, $R_1^2(\kappa) \leq a < b \leq R_1^1(\kappa)$.

Therefore, if $R_1(\kappa) \in [a, b]$, then $G(R_1(\kappa), \kappa) < 0$. On the other hand, recall that $\Phi_{R_1} < 0$. Thus the following inequalities hold: $\forall \kappa \in [\kappa_F, \kappa_S]$,\[\Phi(R_1^1(\kappa), \kappa) \leq \Phi(b, \kappa) < \Phi(a, \kappa) \leq \Phi(R_1^2(\kappa), \kappa).\]

However, since $\Phi(b, \kappa_F) < \Phi(a, \kappa_S)$ by assumption, we have, $\forall \kappa \in \Lambda$,\[\Phi(R_1^1(\kappa), \kappa) < \Phi(b, \kappa) < \Phi(b, \kappa_F) < \Phi(a, \kappa_S) < \Phi(a, \kappa) \leq \Phi(R_1^2(\kappa), \kappa).\]

Therefore, $M \in \Upsilon$. Moreover since in equilibrium $M = \Phi(R_1(\kappa), \kappa)$ and

\[\Phi(b, \kappa_S) < \Phi(b, \kappa_F) < M = \Phi(R_1(\kappa), \kappa) < \Phi(a, \kappa_S) < \Phi(a, \kappa_F),\]

we must have $a < R_1(\kappa) < b$, for all $\kappa \in [\kappa_F, \kappa_S]$.

Now we are ready to prove Proposition 6.11. Note that

\[
R_1^S - R_1^F + \kappa^S - \kappa^F = - \int_{\Lambda} \left( \frac{\Phi_{\kappa}(R_1(\kappa), \kappa)}{\Phi_{R_1}(R_1(\kappa), \kappa)} - 1 \right) d\kappa.
\]

Since $\Phi_{R_1} < 0$, this equation implies that if $\Phi_\kappa(R_1(\kappa), \kappa) - \Phi_{R_1}(R_1(\kappa), \kappa) < 0$ for all $\kappa = [\kappa_F, \kappa_S]$, then $I^S > I^F$, i.e., the second best investment level is higher than the first best. Note that

\[
\Phi_{\kappa}(R_1(\kappa), \kappa) - \Phi_{R_1}(R_1(\kappa), \kappa) = \frac{(\gamma_A + \gamma_P)}{\gamma^A R_1(\kappa) + \kappa^2} G(R_1(\kappa), \kappa).
\]

Let

\[
\Delta := \left(2\kappa - \frac{\gamma_P \gamma_A^2}{4(\gamma_P + \gamma_A)} \right)^2 - 4\kappa^2 = \left(\frac{\gamma_P \gamma_A^2}{4(\gamma_P + \gamma_A)} \right) \left(\frac{\gamma_P \gamma_A^2}{4(\gamma_P + \gamma_A)} - 4\kappa\right)
\]

Part 1: Since $\Delta < 0$ for all $\kappa \in \Lambda$, $G(R_1, \kappa) > 0$ for all $\kappa$, and thus $\Phi_\kappa(R_1(\kappa), \kappa) - \Phi_{R_1}(R_1(\kappa), \kappa) > 0$ for all $\kappa \in \Lambda$, i.e., the first-best production investment level is greater than the second-best.

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Part 2: By assumption, $\Delta > 0$ for all $\kappa$. Note that the roots of equation $G(R_1, \kappa) = 0$ are as follows:

$$
\bar{R}_1^1(\kappa) = \frac{1}{2} \left\{ \frac{\gamma_p \gamma A A^2}{4(\gamma_p + \gamma A)} - 2\kappa + \sqrt{\Delta} \right\},
$$

$$
\bar{R}_1^2(\kappa) = \frac{1}{2} \left\{ \frac{\gamma_p \gamma A A^2}{4(\gamma_p + \gamma A)} - 2\kappa - \sqrt{\Delta} \right\}.
$$

Then, $0 < \bar{R}_1^2(\kappa) < \bar{R}_1^1(\kappa)$. Thus, if $G(R_1(\kappa), \kappa) < 0$ for all $\kappa \in \Lambda$, then $I^S > I^F$. This can be achieved if for each $\kappa \in \Lambda$, $R_1(\kappa)$ lies in between $\bar{R}_1^2(\kappa)$ and $\bar{R}_1^1(\kappa)$. However, recall $\Phi_{R_1} < 0$, and the market-clearing identity, $\Phi(R_1(\kappa), \kappa) \equiv M$ for all $\kappa$. Since $M \in \Upsilon$ by assumption, we have $\Phi(R_1^1(\kappa), \kappa) < \Phi(R_1(\kappa), \kappa) = M < \Phi(R_1^2(\kappa), \kappa)$, which implies that $R_1(\kappa)$, for all $\kappa \in \Lambda$, lies in between $\bar{R}_1^2(\kappa)$ and $\bar{R}_1^1(\kappa)$. 

\[\begin{align*}
\text{J Proof of Proposition 6.12} \\
\text{By applying Propositions (4.1) and (5.3) to our linear-quadratic case, we have the first- and second-best market prices of risk are given as follows.} \\
\theta^F &= \frac{\gamma A \gamma p \sigma D A}{2(\gamma A + \gamma p)} = \frac{\gamma A \gamma p \sigma D}{(\gamma A + \gamma p)} I^F \\
\theta^S &= \frac{\gamma p \gamma A \sigma D A}{2 \left[ \frac{1}{\sigma_D^2} + (\gamma A + \gamma p) \right] (R_1^S + \kappa S)} = \frac{\gamma p \gamma A \sigma D}{\left[ \frac{1}{\sigma_D^2} + (\gamma A + \gamma p) \right]} I^S \\
\nu_0^F &= \frac{\theta^2}{\gamma p R_1 S_0} = \frac{(\gamma A \gamma p \sigma D)^2}{\gamma p R_1^F (\gamma A + \gamma p)^2} = \frac{\gamma p \gamma A \sigma D^2}{R_1^F (\gamma A + \gamma p)^2} \\
\nu_0^S &= \frac{\theta^2}{\gamma p R_1 S_0} = \frac{(\gamma A \gamma p \sigma D)^2}{\gamma p R_1^S \left( \frac{1}{\sigma_D^2} + (\gamma A + \gamma p) \right)^2} = \frac{\gamma p \gamma A \sigma D^2}{R_1^S \left( \frac{1}{\sigma_D^2} + (\gamma A + \gamma p) \right)^2} \\
\text{Thus, the assertion follows.}
\end{align*}\]
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