Equilibrium Portfolio Strategies in the Presence of Sentiment Risk and Excess Volatility

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Main motivation: Excessive volatility in the stock market

- Robert Shiller (1981): Actual stock price behavior compared to present discounted value of dividends
  - Stock prices are “too volatile” to be justified by subsequent dividends
  - Volatility is excessive relative to fundamentals
- How can one exploit this form of market inefficiency?
  - if it cannot be exploited easily, it will survive a long time
Our objective

- Suppose that a financial market is deemed to be affected by fluctuations in market “sentiment”,
  - so that sentiment volatility causes prices to be more volatile than what would be justified by dividend volatility alone.

- Suppose further that a Bayesian, intertemporally optimizing investor trades in that market.
  - We would like to know what investment policy this person will undertake under market equilibrium

- We build a general-equilibrium model of a financial market in which a subpopulation of investors trades on "sentiment" and generates excess volatility.
  - In our model, some investors (Group A) are “overconfident” in the sense that they give too much credence to a public information signal
The “fundamental” and its dynamics

**Assumption 1:** Under the beliefs of Group B, the process for aggregate dividends is driven by the following pair of stochastic differential equations, which defines a Markovian system in two state variables, \( \{ \delta_t, \hat{f}_t^B \} \):

\[
\frac{d\delta_t}{\delta_t} = \hat{f}_t^B \, dt + \sigma_{\delta} \, dW_{\delta,t}^B,
\]

\[
d\hat{f}_t^B = -\zeta \left( \hat{f}_t^B - \bar{f} \right) \, dt + \frac{\gamma_{\delta}^B}{\sigma_{\delta}} \, dW_{\delta,t}^B,
\]

where \( W_{\delta}^B \) is a one-dimensional process that is Brownian under the probability measure that reflects the expectations of Group B.
The “sentiment” and its dynamics

Our economy is a heterogeneous-expectations economy, where the belief of Group A about the expected growth rate of aggregate dividends differs from that of Group B.

**Assumption:** Group A believes that the expected growth rate of aggregate dividends is equal to:

$$f^A_t = f^B_t - \hat{g}_t.$$  

Hence $\hat{g}_t$ is the “difference of opinion”. The dynamics of the difference of opinion are specified to follow an Ornstein-Uhlenbeck process that mean reverts ($\psi > 0$) to zero:

$$d\hat{g}_t = -\psi \hat{g}_t dt + \sigma_{\hat{g},\delta} dW^B_{\delta,t} + \sigma_{\hat{g},s} dW^B_{s,t},$$  

(1)

where $W^B_s$ is a second one-dimensional process, independent of $W^\delta$ that is Brownian under the probability measure that reflects the expectations of Group B.
The “sentiment” and its dynamics

Group A differs from Group B in its beliefs about the aggregate dividends process. Group A’s probability beliefs at time $t$ are represented by a change of measure $\eta$, where $\{\eta_t\}$ is a strictly positive martingale process. For any event $e_u$ belonging to the $\sigma$-algebra of time $u$, we have:

$$E^A_t [1_{e_u}] = E^B_t \left[ \frac{\eta_u}{\eta_t} 1_{e_u} \right].$$

**Lemma 1**: (Girsanov)

$$\frac{d\eta_t}{\eta_t} = -\hat{g}_t \sigma_{\delta}^{-1} dW^B_{\delta,t}. \quad (2)$$

We refer to the pair of equations (1,2) as the “sentiment system”. 
Two effects

One determines the *volatility of sentiment*, the other the *volatility of volatility*

\[
\begin{align*}
\delta & & dW^B_\delta & & dW^B_s \\
f^B & \left[ \begin{array}{cc}
\delta & \delta & \sigma_\delta & > & 0 & 0 \\
\gamma^B & \sigma_\delta & > & 0 & 0 \\
-\eta & \frac{\hat{g}}{\sigma_\delta} & & 0 \\
\sigma_{\hat{g},\delta} & \geq & 0 & \sigma_{\hat{g},s} & \leq & 0 \\
\end{array} \right].
\end{align*}
\]

The difference of opinion \( \hat{g} \) scales the diffusion of \( \eta \), which implies that \( \eta \) has a diffusion that can take large positive or negative values. Sentiment or heterogeneity of beliefs, \( \eta \), is volatile. This first effect is cumulative over time, or long term.

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1. The difference of opinion \( \hat{g} \) scales the diffusion of \( \eta \), which implies that \( \eta \) has a diffusion that can take large positive or negative values. Sentiment or heterogeneity of beliefs, \( \eta \), is volatile. This first effect is cumulative over time, or long term.
Two effects

One determines the volatility of sentiment, the other the volatility of volatility

\[
\begin{bmatrix}
\delta & dW_\delta^B & dW_s^B \\
\gamma^B / \sigma_\delta & > 0 & 0 \\
-\eta \frac{\hat{g}}{\sigma_\delta} & 0 \\
\sigma_{\hat{g},\delta} & \geq 0 & \sigma_{\hat{g},s} \leq 0 \\
\end{bmatrix}
\]

1. The difference of opinion \( \hat{g} \) scales the diffusion of \( \eta \), which implies that \( \eta \) has a diffusion that can take large positive or negative values. Sentiment or heterogeneity of beliefs, \( \eta \), is volatile. This first effect is cumulative over time, or long term.

2. Second, the difference of opinion, \( \hat{g} \), itself is stochastic. This second effect (the effect of the dynamics of heterogeneous beliefs) is instantaneous or short-term.
the problem of Group $B$ is to maximize the expected utility from lifetime consumption:

$$\sup_c \mathbb{E}^B \int_0^\infty e^{-\rho t} \frac{1}{\alpha} \left( c_t^B \right)^\alpha dt; \quad \alpha < 1,$$

s.t.: $\mathbb{E}^B \int_0^\infty \zeta_t^B c_t^B dt = \bar{\theta}^B \mathbb{E}^B \int_0^\infty \zeta_t^B \delta_t dt$,

Under $B$’s probability measure, the problem of $A$ can be stated as follows:

$$\sup_c \mathbb{E}^B \int_0^\infty \eta_t e^{-\rho t} \frac{1}{\alpha} \left( c_t^A \right)^\alpha dt,$$

s.t.: $\mathbb{E}^B \int_0^\infty \zeta_t^B c_t^A dt = \bar{\theta}^A \mathbb{E}^B \int_0^\infty \zeta_t^B \delta_t dt$.

The first-order condition for consumption in this case is

$$\eta_t e^{-\rho t} \left( c_t^A \right)^{\alpha-1} = \lambda^A \zeta_t^B,$$
Equilibrium

Solving the aggregate resource constraint:

\[
\zeta_t^B(\delta_t, \eta_t) = e^{-\rho t} \left[ \left( \frac{\eta_t}{\lambda^A} \right)^{\frac{1}{1-\alpha}} + \left( \frac{1}{\lambda^B} \right)^{\frac{1}{1-\alpha}} \right]^{1-\alpha} \delta_t^{\alpha-1},
\]

and, therefore:

\[
c_t^A = \omega(\eta_t) \delta_t; \quad c_t^B = (1 - \omega(\eta_t)) \delta_t,
\]

where:

\[
\omega(\eta_t) \triangleq \frac{\left( \frac{\eta_t}{\lambda^A} \right)^{\frac{1}{1-\alpha}}}{\left( \frac{\eta_t}{\lambda^A} \right)^{\frac{1}{1-\alpha}} + \left( \frac{1}{\lambda^B} \right)^{\frac{1}{1-\alpha}}},
\]
Prices of securities will be expected values of payoff $\times \xi_t^B(\delta_t, \eta_t)$.

The curvature of $\xi_t^B(\delta_t, \eta_t)$ with respect to $\eta$ implies that heterogeneity of expectations reduces asset prices when risk aversion $> 1$.

Economic reason in terms of income and substitution effects?
We find an explicit solution for the prices of all claims

This is the first general-equilibrium utilization of the exponential linear-quadratic framework

**Proposition 1:** The moment-generating-function for the joint distribution of $\delta$ and $\eta$ at maturity $u$ under the measure of Group $B$ is given by

$$
\mathbb{E}_B^{\hat{f}_B, \hat{g}} \left[ \left( \frac{\delta_u}{\delta} \right)^\varepsilon \left( \frac{\eta_u}{\eta} \right)^\chi \right] = H_f \left( \hat{f}_B^B, t, u; \varepsilon \right) \times H_g \left( \hat{g}, t, u; \varepsilon, \chi \right),
$$

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\[ H_f \left( \hat{f}^B, u, t; \varepsilon \right) = \exp \left\{ \varepsilon \left[ \bar{f} (u - t) + \frac{1}{\zeta} \left( \bar{f}^B - \bar{f} \right) \left[ 1 - e^{-\zeta (u-t)} \right] \right] \right. \\
+ \frac{1}{2} \varepsilon (\varepsilon - 1) \sigma_\delta^2 (u - t) + \frac{\varepsilon^2 \gamma^B}{2 \zeta^2} \left[ 1 - e^{-\zeta (u-t)} \right]^2 \\
\left. + \frac{\varepsilon^2 \sigma_f^2}{4 \zeta^3} \left[ 2 \zeta (u - t) - 3 + 4e^{-\zeta (u-t)} - e^{-2\zeta (u-t)} \right] \right\}, \]

\[ H_g \left( \hat{g}, u, t; \varepsilon, \chi \right) = \exp \left\{ A_1 (\chi; u - t) + \varepsilon^2 A_2 (\chi; u - t) \\
+ \varepsilon \hat{g} B (\chi; u - t) + \hat{g}^2 C (\chi; u - t) \right\}. \]

where the functions \( A_1, A_2, B \) and \( C \) are defined in the proof.
Special case in which risk aversion is an integer, \((1 - \alpha) \in \mathbb{N}\), which excludes the cases of risk aversion smaller than 1 (that is, \(\alpha > 0\)).

The measure (3) can be written in an alternative way by expanding the bracket into an exact finite sum by virtue of the binomial formula:

\[
\left[ \left( \frac{\eta_u}{\lambda^A} \right)^{\frac{1}{1-\alpha}} + \left( \frac{1}{\lambda^B} \right)^{\frac{1}{1-\alpha}} \right]^{1-\alpha} = \frac{1}{\lambda^B} \sum_{j=0}^{1-\alpha} C_{1-\alpha}^j \left( \frac{\eta_u \lambda^B}{\lambda^A} \right)^{\frac{j}{1-\alpha}},
\]

where \(C_{1-\alpha}^j\) denotes the binomial coefficient \(\frac{(1-\alpha)!}{j!(1-\alpha-j)!}\).

Equilibrium prices are sums of exponential linear quadratic functions.
Diffusion matrix of securities (exposures to shocks)

\[
\begin{bmatrix}
\text{diff} F \\
\text{diff} P
\end{bmatrix} = \begin{bmatrix}
\frac{\partial F}{\partial \delta} & \frac{\partial F}{\partial \delta f} & \frac{\partial F}{\partial \eta} & \frac{\partial F}{\partial \hat{g}} \\
\frac{\partial P}{\partial \delta} & \frac{\partial P}{\partial \delta f} & \frac{\partial P}{\partial \eta} & \frac{\partial P}{\partial \hat{g}}
\end{bmatrix} \begin{bmatrix}
\frac{\delta \sigma_\delta}{\gamma^B} & 0 \\
\frac{\gamma^B}{\sigma_\delta} & 0 \\
-\eta \frac{\hat{g}}{\sigma_\delta} & 0 \\
\frac{\gamma^B - \gamma^A}{\sigma_\delta} & -\phi \sigma_f
\end{bmatrix}.
\]

It will be important to separate the diffusion components that arise from the movements of the difference of opinion. We define the diffusion arising from the first three state variables as:

\[
\begin{bmatrix}
\text{diff}_3 F \\
\text{diff}_3 P
\end{bmatrix} \triangleq \begin{bmatrix}
\frac{\partial F}{\partial \delta} & \frac{\partial F}{\partial \delta f} & \frac{\partial F}{\partial \eta} & \frac{\partial F}{\partial \hat{g}} \\
\frac{\partial P}{\partial \delta} & \frac{\partial P}{\partial \delta f} & \frac{\partial P}{\partial \eta} & \frac{\partial P}{\partial \hat{g}}
\end{bmatrix} \begin{bmatrix}
\frac{\delta \sigma_\delta}{\gamma^B} & 0 \\
\frac{\gamma^B}{\sigma_\delta} & 0 \\
-\eta \frac{\hat{g}}{\sigma_\delta} & 0
\end{bmatrix}.
\]

Clearly:

\[
\begin{bmatrix}
\text{diff} F \\
\text{diff} P
\end{bmatrix} = \begin{bmatrix}
\text{diff}_3 F \\
\text{diff}_3 P
\end{bmatrix} + \begin{bmatrix}
\frac{\partial F}{\partial \hat{g}} & \frac{\partial F}{\partial \hat{g}} \\
\frac{\partial P}{\partial \hat{g}} & \frac{\partial P}{\partial \hat{g}}
\end{bmatrix} \begin{bmatrix}
\frac{\gamma^B - \gamma^A}{\sigma_\delta} & -\phi \sigma_f
\end{bmatrix}.
\]
Diffusion matrix of securities (exposures to shocks)
Diffusion matrix of securities (exposures to shocks)

We draw three inferences from this figure:

1. The effect of $\phi$, except through $\hat{g}$, is small:

$$
\begin{bmatrix}
\frac{\text{diff} F}{F} \\
\frac{\text{diff} P}{P}
\end{bmatrix}_{\phi=0} \approx \begin{bmatrix}
\frac{\text{diff}_3 F}{F} \\
\frac{\text{diff}_3 P}{P}
\end{bmatrix}_{\phi \neq 0}.
$$

2. The changes in the diffusion vector of assets arising from overconfidence are entirely the result of the $\hat{g}$ component.

3. Equity is mostly positively exposed to the realized innovation in the fundamental, $dW^B_\delta$, whereas the bond is negatively exposed.
Overconfidence induces excess volatility

Equity volatility

Bond volatility

Equity volatility

Bond volatility
Portfolio strategy of Group B (proper beliefs)

\[ F^{B,T}(\delta, \hat{f}^B, \eta, \hat{g}, t) = E^B_{\delta, \eta, \hat{f}^B, \hat{g}} \left[ \frac{\zeta^{B}_T}{\zeta^B_t} c^B \right] \]

We need to solve for \( \theta^T \) (a 1 \times 2 vector) the following set of two equations:

\[
\begin{bmatrix}
F^B_t \frac{\partial F^B_t}{\partial \hat{f}^B} & \frac{\partial F^B_t}{\partial \eta} & \frac{\partial F^B_t}{\partial \hat{g}} \\
F^B_t & \frac{\partial F^B_t}{\partial \eta} & \frac{\partial F^B_t}{\partial \hat{g}}
\end{bmatrix}
\begin{bmatrix}
\sigma_\delta \\
\gamma^B_\sigma \\
\gamma^B_\sigma - \gamma^A_\sigma \\
-\eta \frac{\hat{g}}{\sigma_\delta}
\end{bmatrix}
= \theta^T
\begin{bmatrix}
\frac{\sigma_\delta}{\sigma_\delta} & 0 \\
\gamma^B_\sigma & 0 \\
\gamma^B_\sigma - \gamma^A_\sigma & -\phi \sigma_f \\
-\eta \frac{\hat{g}}{\sigma_\delta} & 0
\end{bmatrix}
\]
Portfolio choice

When $\phi = 0$, the portfolio is indeterminate. We consider the "symmetric benchmark":

$$
\begin{bmatrix}
\frac{F^B_t}{F_t} \\
\frac{\partial F^B_t}{\partial \theta} \\
\frac{\partial P_t}{\partial \theta}
\end{bmatrix}_{\phi=0}
$$

**Proposition 2:** For as long as $\phi \neq 0$, the portfolio choice is independent of the specific value of $\phi$ except through the value functions $F$, $P$ and $F^B$, and the solution is:

$$
\theta_F = \begin{vmatrix}
\text{diff}_3 F^B_t & \frac{\partial F^B_t}{\partial \theta} \\
\text{diff}_3 F_t & \frac{\partial F_t}{\partial \theta} \\
\text{diff}_3 P_t & \frac{\partial P_t}{\partial \theta}
\end{vmatrix}, \quad
\theta_P = \begin{vmatrix}
\text{diff}_3 F^B_t & \frac{\partial F^B_t}{\partial \theta} \\
\text{diff}_3 F_t & \frac{\partial F_t}{\partial \theta} \\
\text{diff}_3 P_t & \frac{\partial P_t}{\partial \theta}
\end{vmatrix},
$$

where $\theta_F$ is the number of units of equity demanded and $\theta_P$ the number of units of the bond.
Portfolio choice

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When $\hat{g} = 0$, Group $B$ holds fewer units of equity than the number corresponding to their share of wealth:

$$\theta_F < \frac{F_t^{B,T}}{F_t^T}$$

Risk averse investors with the proper beliefs of Group $B$ are deterred by the presence of the overconfident traders, whose difference of opinion is a source of risk in their eyes.

The demand schedule for equity is upward sloping as a function of $\hat{g}$. Even though Group $B$ are driven away by the presence of the overconfident traders, they overcome their fear when they are very optimistic about future growth.

$B'$s demand schedule for bonds is downward sloping. Investing in equity also exposes them to future changes in discount rates and in the beliefs of others. Group $B$ uses bonds to hedge these other risks.
On long-run predictability

- Malliavin derivative measures change in a path-dependent function implied by a small change in initial value of the underlying Brownian motion.
- Allows for a clean & insightful interpretation of the results, and in particular, to distinguish between instantaneous effects and long-term effects.

**Lemma 3**: The Malliavin derivatives for the four state variables are:

\[
\begin{align*}
\mathcal{D}_t^B \hat{f}_u^B &= e^{-\zeta(u-t)} \left[ \frac{\gamma^B}{\sigma_\delta} 0 \right], \\
\mathcal{D}_t^B \hat{g}_u &= e^{-\psi(u-t)} \left[ \frac{\gamma^B - \gamma^A}{\sigma_\delta} - \phi \sigma_f \right], \\
\mathcal{D}_t^B \delta_T &\delta_T = \left[ \sigma_\delta 0 \right] + \frac{1}{\zeta} \left( 1 - e^{-\zeta(T-t)} \right) \left[ \frac{\gamma^B}{\sigma_\delta} 0 \right], \\
\mathcal{D}_t^B \eta_T &\eta_T = \left[ \frac{-\hat{g}_t}{\sigma_\delta} \frac{-\hat{g}_t}{\sigma_s} \right] - \int_t^T e^{-\psi(u-t)} \left( \frac{\hat{g}_u}{\sigma^2_\delta} du + \frac{dW_{\delta,u}^B}{\sigma_\delta} \right) \\
& \quad \times \left[ \frac{\gamma^B - \gamma^A}{\sigma_\delta} - \phi \sigma_f \right].
\end{align*}
\]
Lemma 4: In equilibrium, the market prices of risk in the eyes of Group B is:

\[
\frac{D_t^B \xi_B^t}{\xi_B^t} = -\kappa_t^B = (\alpha - 1) \frac{D_t^B \delta_t}{\delta_t} + \omega(\eta_t) \frac{D_t^B \eta_t}{\eta_t}
\]

\[
= - \begin{bmatrix} (1 - \alpha) \sigma_\delta \\ 0 \end{bmatrix} - \hat{g}\omega(\eta) \begin{bmatrix} \frac{1}{\sigma_\delta} \\ 0 \end{bmatrix}.
\]

- The prices of risk \( \kappa^B \) contains an instantaneous premium for the output shock \( W_\delta \) but no instantaneous premium for the signal shock \( W_s \).
- If there is no difference of opinion (\( \hat{g} = 0 \)), the prices of risk \( \kappa^B \) includes only the traditional reward for fundamental risk \( (1 - \alpha) \sigma_\delta \).
- As soon as there is a difference of opinion, investors realize that “sentiment” will fluctuate randomly in response to output shocks. Hence, they start charging a premium for the risk arising from the vagaries of others.
Long-run pricing of risk

\[
\frac{D_t^B \zeta_T^B}{\zeta_T^B} = (\alpha - 1) \frac{D_t^B \delta_T}{\delta_T} + \omega(\eta_T) \frac{D_t^B \eta_T}{\eta_T}.
\]

The long-run return has two components:

1. the first arising from the fluctuations in output and
2. the second arising from the vagaries of the overconfident population.

- The term, \( \omega(\eta_T) \frac{D_t^B \eta_T}{\eta_T} \), which would not be present in a market without overconfident investors (\( \omega = 0 \)), is a predictable component and it modifies the long-run behavior of returns.
Portfolio strategy in terms of short-run and long-run returns

**Proposition 3:** The Malliavin derivative of $B$’s wealth reveals the portfolio strategy they pursue:

$$
D^B_t F^{B,T}_t = -\frac{1}{1-\alpha} F^{B,T}_t \frac{D^B_t \xi^B_t}{\xi^B_t} + \frac{\alpha}{\alpha - 1} E^B_t \left[ \frac{c^B_T \xi^B_T}{\xi^B_T} \left( \frac{D^B_t \xi^B_t}{\xi^B_T} - \frac{D^B_t \xi^B_t}{\xi^B_t} \right) \right]
$$

$$
\frac{D^B_t F^{B,T}_t}{F^{B,T}_t} = \frac{\kappa^B_t}{1-\alpha} + \frac{\alpha}{\alpha - 1} \frac{1}{F^{B,T}_t} E^B_t \left[ \frac{c^B_T \xi^B_T}{\xi^B_T} \left( \frac{D^B_t \xi^B_t}{\xi^B_T} - \frac{D^B_t \xi^B_t}{\xi^B_t} \right) \right]
$$

*myopic* \hspace{2cm} *intertemporal*

The first term is a myopic portfolio and the second one an intertemporal hedge, which anticipates future returns.
Portfolio strategy in terms of short-run and long-run returns

In particular, one component of the last term is:

\[
\frac{\alpha}{\alpha - 1} \frac{1}{F_t^{B,T}} \mathbb{E}_t B \left[ \sigma_B \frac{\xi_T}{\xi_t} \omega (\eta_T) \left\{ - \int_t^T D_t^B \hat{g}_u \left[ \frac{\hat{g}_u}{\sigma^2} du + \frac{dW_{\delta,u}^B}{\sigma_{\delta}} \right] \right\} \right]
\]

\[
= \frac{\partial F_t^{B,T}}{\partial \hat{g}} \left[ \frac{\gamma^B - \gamma^A}{\sigma_{\delta}} - \phi \sigma_f \right]
\]

The more “strategic” exploitation of the long-run predictability created by overconfident investors is imbedded in the intertemporal hedge. Group B knows that

- its share of consumption will fluctuate,
- that it will revise its expectations of growth,
- that the other group also will and that it will do so in a manner different from theirs.
Two benefits from the model

1. First, of **practical use to hedge funds** who play the price-convergence game and in so doing expose themselves to “market sentiment” risk.
   - Hedge funds need a model of the equilibrium stochastic process which describes how sentiment will drive price spreads.
   - We caution, however, that the gains will accumulate at a slow pace.

2. Second, the model parsimoniously combines
   - the **technical virtues** of continuous-time, rational-expectations equilibrium asset pricing models (including use of martingale approach)
   - with a **single, well-defined, almost axiomatic deviation** from the case where all agents have the proper beliefs. In this way, it has allowed us to analyze the equilibrium consequences of that specific deviation.
   - We hope that this model and similar models obtained by this method can become workhorses in **development of behavioral equilibrium theory**.