Comparative Statics of General Equilibrium
Asset Prices

Theodoros M. Diasakos

Collegio Carlo Alberto

November 8, 2010
The Theoretical Issue

» Dynamic Completeness in Continuous-time: surprisingly little has been established theoretically, even for potentially dynamically-complete markets ($N = K$).
The Theoretical Issue

- Dynamic Completeness in Continuous-time: surprisingly little has been established theoretically, even for potentially dynamically-complete markets ($N = K$).
  - Single agent: equilibrium exists but is it dynamically-complete ($K > 1$)?
  - Many agents: For $K > 1$, we compute the candidate equilibrium price process explicitly and then verify that it is dynamically-complete. Yet, this can be done only for a very small set of parameter values without ruling out the possibility that it is of zero measure.
The Theoretical Issue

- Dynamic Completeness in Continuous-time: surprisingly little has been established theoretically, even for potentially dynamically-complete markets \((N = K)\).
  - Single agent: equilibrium exists but is it dynamically-complete \((K > 1)\)?
  - Many agents: For \(K > 1\), we compute the candidate equilibrium price process explicitly and then verify that it is dynamically-complete. Yet, this can be done only for a very small set of parameter values without ruling out the possibility that it is of zero measure.
  - Problem even if, say, all agents are CRRA with different RA parameters. The representative agent is not CRRA.
The Theoretical Issue

- Dynamic Completeness in Continuous-time: surprisingly little has been established theoretically, even for potentially dynamically-complete markets ($N = K$).
  - Single agent: equilibrium exists but is it dynamically-complete ($K > 1$)?
  - Many agents: For $K > 1$, we compute the candidate equilibrium price process explicitly and then verify that it is dynamically-complete. Yet, this can be done only for a very small set of parameter values without ruling out the possibility that it is of zero measure.
  - Problem even if, say, all agents are CRRA with different RA parameters. The representative agent is not CRRA.

- “Contagion:” Correlations across asset-returns that cannot be explained by covariances between their respective payoffs alone. Cochrane et al. (2008), Martin (2008).
The Empirical Issue


- Correlations across asset returns arise here endogenously and are stochastic, even though the covariance coefficients of asset-payoffs are constant.
The Empirical Issue


- Correlations across asset returns arise here endogenously and are stochastic, even though the covariance coefficients of asset-payoffs are constant.


- Asset prices are correlated here even when the asset payoffs are independent.
General Structure

- Single-good, pure-exchange, continuous-time economy with identical agents and Lucas’ trees.
General Structure

- Single-good, pure-exchange, continuous-time economy with identical agents and Lucas’ trees.

- Uncertainty is described by a standard $K$-dimensional Brownian motion $\beta = (\beta_1, \ldots, \beta_K)^T \in \mathbb{R}^K$ and the filtration $\{\mathcal{F}_t : t \in [0, T]\}$ it generates on a probability space $(\Omega, \mathcal{F}, \mu)$. 
General Structure

- Single-good, pure-exchange, continuous-time economy with identical agents and Lucas’ trees.

- Uncertainty is described by a standard $K$-dimensional Brownian motion $\beta = (\beta_1, ..., \beta_K)^T \in \mathbb{R}^K$ and the filtration $\{\mathcal{F}_t : t \in [0, T]\}$ it generates on a probability space $(\Omega, \mathcal{F}, \mu)$.

- Trade and consumption occur over the compact time interval $[0, T]$ which is endowed with a measure $\lambda$ such that it agrees with the Lebesgue measure on $[0, T)$ and $\lambda(\{T\}) = 1$. 

General Structure

- Single-good, pure-exchange, continuous-time economy with identical agents and Lucas’ trees.
- Uncertainty is described by a standard $K$-dimensional Brownian motion $\beta = (\beta_1, \ldots, \beta_K)^T \in \mathbb{R}^K$ and the filtration $\{\mathcal{F}_t : t \in [0, T]\}$ it generates on a probability space $(\Omega, \mathcal{F}, \mu)$.

- Trade and consumption occur over the compact time interval $[0, T]$ which is endowed with a measure $\lambda$ such that it agrees with the Lebesgue measure on $[0, T)$ and $\lambda(\{T\}) = 1$.
- Available for trade are $N + 1$ securities, each specified by the respective dividend process (in units of consumption).
Asset Structure

A zero-coupon bond $B$ (in zero net supply). Its payoff is given by $D_0 : \Omega \times [0, T] \rightarrow \mathbb{R}_+$ with

$$D_0(\omega, T) = 1 \quad D_0(\omega, t) = 0 \quad \forall t \in [0, T)$$
Asset Structure

- A zero-coupon bond $B$ (in zero net supply). Its payoff is given by $D_0 : \Omega \times [0, T] \rightarrow \mathbb{R}_+$ with
  
  $$D_0(\omega, T) = 1 \quad D_0(\omega, t) = 0 \quad \forall t \in [0, T)$$

- $N \leq K$ risky securities (in net supply of one unit). Each pays off only at $T$ and the dividend follows a geometric Brownian motion. Taking $\Sigma$ to be a $N \times K$ matrix of factor loadings with $\sigma_n^T$ its $n$th row,
  
  $$D_n(\omega, T) = e^{\mu_n T + \sigma_n^T \beta(\omega, T)} : \mu_n \in \mathbb{R}_+, \sigma_n \in \mathbb{R}^K$$
Preferences & Endowment

The representative agent has an additively-separable, time-independent utility function,

\[ U(c) = \mathbb{E}_\mu \left[ \int_0^T v(c_t) \, dt + u(c_T) \right] \]

for a measurable consumption function \( c : \Omega \times [0, T] \to \mathbb{R}^+ \) and twice continuously-differentiable functions \( v, u : \mathbb{R}^+ \to \mathbb{R} \) with \( v', u' > 0, \ v'', u'' < 0 \).

Her endowment process, \( e : \Omega \times [0, T] \to \mathbb{R}_+ \), is constant except for the terminal period \( T \):

\[ e(\omega, T) = \rho(\beta(\omega, T)) \quad e(\omega, t) = 1 \ \forall (\omega, t) \in \Omega \times [0, T) \]

for some continuous function \( \rho : \mathbb{R}^K \to \mathbb{R}_+ \).
Raimondo (2005) shows that there exists an equilibrium price process for this economy, when markets are potentially dynamically-complete \((N = K)\) and when they are necessarily dynamically-incomplete \((N < K)\).
Raimondo (2005) shows that there exists an equilibrium price process for this economy, when markets are potentially dynamically-complete \((N = K)\) and when they are necessarily dynamically-incomplete \((N < K)\).

It can be obtained in terms of the agent’s utility function (in particular, her attitude towards risk), her endowment in the terminal period \(T\), and the current realization of the sources of uncertainty (as given by the vector \(\beta\)).
Raimondo (2005) shows that there exists an equilibrium price process for this economy, when markets are potentially dynamically-complete \((N = K)\) and when they are necessarily dynamically-incomplete \((N < K)\).

It can be obtained in terms of the agent’s utility function (in particular, her attitude towards risk), her endowment in the terminal period \(T\), and the current realization of the sources of uncertainty (as given by the vector \(\beta\)).
The equilibrium price processes are defined by

\[
P_n(\omega, t) = \int \cdots \int_{\mathbb{R}^K} u'(W(\omega, t, x)) e^{\mu n T + \sigma_n^T (\beta(\omega, t) + \sqrt{T - t} x)} d\Phi(x)
\]

\[
P_0(\omega, t) = \int \cdots \int_{\mathbb{R}^K} u'(W(\omega, t, x)) d\Phi(x)
\]

\[
P_c(\omega, t) = \begin{cases} 
v'(1) & t \in [0, T) \\
v'(W(\omega, T, 0)) & t = T \end{cases}
\]

where

\[
W(\omega, t, x) = \rho \left( \beta(\omega, t) + \sqrt{T - t} x \right)
\]

\[
+ \sum_{i=1}^{N} e^{\mu_i T + \sigma_i^T (\beta(\omega, t) + \sqrt{T - t} x)}
\]

is the terminal-period wealth (in units of consumption).
Comparative Statics of Relative Prices

I examine the comparative statics, with respect to changes in the Brownian vector $\beta(\omega, t): t \in [0, T)$, of the relative equilibrium price of the $n$th stock:

$$\frac{P_n(\omega, t)}{P_0(\omega, t)} = e^{\mu_n T + \sigma_n^T \beta(\omega, t)} \frac{\mathbb{E}_x \left[ u'(F(\omega, t, x)) e^{\sqrt{T-t} \sigma_n^T x} \right]}{\mathbb{E}_x \left[ u'(F(\omega, t, x)) \right]}$$

where the expectations are taken with respect to $x \sim \mathcal{N}(0, I_K)$. 
Comparative Statics of Relative Prices

I examine the comparative statics, with respect to changes in the Brownian vector $\beta(\omega, t) : t \in [0, T)$, of the relative equilibrium price of the $n$th stock:

$$\frac{P_n(\omega, t)}{P_0(\omega, t)} = e^{\mu_n T + \sigma_n^T \beta(\omega, t)} \frac{\mathbb{E}_x \left[ u'(F(\omega, t, x)) e^{\sqrt{T-t} \sigma_n^T x} \right]}{\mathbb{E}_x [u'(F(\omega, t, x))]}$$

where the expectations are taken with respect to $x \sim \mathcal{N}(\mathbf{0}, I_K)$.

That is, we study the quantity

$$\frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} = \frac{\partial}{\partial \beta_k(\omega, t)} \left( \frac{P_n(\omega, t)}{P_0(\omega, t)} \right)$$

$$= \frac{1}{P_0(\omega, t)} \left[ \frac{\partial P_n(\omega, t)}{\partial \beta_k(\omega, t)} - \frac{P_n(\omega, t)}{P_0(\omega, t)} \frac{\partial P_0(\omega, t)}{\partial \beta_k(\omega, t)} \right]$$
Why relative prices?

- Absolute prices are given in terms of marginal utility and measured in units of utility. In practice, marginal utilities are not observable and securities are priced with respect to a numeraire, such as dollars.
Why relative prices?

- Absolute prices are given in terms of marginal utility and measured in units of utility. In practice, marginal utilities are not observable and securities are priced with respect to a numeraire, such as dollars.

- Here, we take consumption to be the numeraire: $P_c(\omega, t) \equiv 1$ for all $t \in [0, T)$. 
Why relative prices?

- Absolute prices are given in terms of marginal utility and measured in units of utility. In practice, marginal utilities are not observable and securities are priced with respect to a numeraire, such as dollars.

- Here, we take consumption to be the numeraire: $P_c(\omega, t) \equiv 1$ for all $t \in [0, T)$.

- The choice of numeraire is essentially arbitrary since the equilibrium market-clearing condition depends only on the relative prices of the securities and consumption, node-$(\omega, t)$ by node-$(\omega, t)$, for $t \neq s$, n a given Brownian path; not across paths.
Why changes w.r.t. $\beta(\omega, t)$?

- The Brownian vector, $\beta(\omega, t)$, depicts the (unobservable) sources of fundamental uncertainty in this economy in terms of what affects the payoffs of risky assets.
Why changes w.r.t. $\beta(\omega, t)$?

- The Brownian vector, $\beta(\omega, t)$, depicts the (unobservable) sources of fundamental uncertainty in this economy in terms of what affects the payoffs of risky assets.

- Central question here is monotonicity. If the relation between $p_n(\beta(\omega, t))$ is monotone, it can be inverted to take us from the price ratio that we observe to the underlying stochastic process that we don’t.
Why changes w.r.t. $\beta(\omega, t)$?

- The Brownian vector, $\beta(\omega, t)$, depicts the (unobservable) sources of fundamental uncertainty in this economy in terms of what affects the payoffs of risky assets.

- Central question here is monotonicity. If the relation between $p_n(\beta(\omega, t))$ is monotone, it can be inverted to take us from the price ratio that we observe to the underlying stochastic process that we don’t. This is fundamental
  - for making predictions, in terms of comparative statics,
  - for checking dynamic completeness, and
  - for empirical analysis.
Why this setting

- Raimondo (2005), Anderson and Raimondo (2005) offer a tractable functional form for the relative price ratio.
Why this setting

- Raimondo (2005), Anderson and Raimondo (2005) offer a tractable functional form for the relative price ratio.
- Nothing pathological about this setting. Standard benchmark in continuous-time finance.
Why this setting

- Raimondo (2005), Anderson and Raimondo (2005) offer a tractable functional form for the relative price ratio.
- Nothing pathological about this setting. Standard benchmark in continuous-time finance.
- The equilibrium value of a security is given by the conditional expectation of its future dividend, valued at the marginal utility of future equilibrium consumption.

\[ E[\text{MU} \times \text{DIVIDEND} | \mathcal{F}_t] \]
Why this setting

- Raimondo (2005), Anderson and Raimondo (2005) offer a tractable functional form for the relative price ratio.
- Nothing pathological about this setting. Standard benchmark in continuous-time finance.
- The equilibrium value of a security is given by the conditional expectation of its future dividend, valued at the marginal utility of future equilibrium consumption.

\[ E[MU \times \text{DIVIDEND} | \mathcal{F}_t] \]

As is well known, this has to be the equilibrium pricing relation in any general equilibrium setting. The comparative statics results obtained here would generalize to other settings; e.g. Cox et al. (1985).
Relative Prices: Overview

\[
\frac{P_n(\omega, t)}{P_0(\omega, t)} = e^{\mu_n T + \sigma_n^T \left( \beta(\omega, t) + \frac{T-t}{2} \sigma_n \right)}
\]

\[
\frac{\mathbb{E}_x \left[ u' \left( W(\omega, t, x + \sqrt{T-t} \sigma_n) \right) \right]}{\mathbb{E}_x \left[ u' \left( W(\omega, t, x) \right) \right]}
\]
Relative Prices: Overview

\[
\frac{P_n(\omega, t)}{P_0(\omega, t)} = e^{\mu_n T + \sigma_n^T \left( \beta(\omega, t) + \frac{(T-t)}{2} \sigma_n \right)}
\]

\[
\mathbb{E}_x \left[ u' \left( W(\omega, t, x + \sqrt{T-t} \sigma_n) \right) \right] \quad \mathbb{E}_x \left[ u' \left( W(\omega, t, x) \right) \right]
\]

Given \( \beta(\omega, t) \), exchanging at time \( t \) one unit of the bond for one unit of the \( n \)th security results in an increase in expected wealth by its expected terminal-period dividend

\[
\mathbb{E}_x \left[ e^{\mu_n T + \sigma_n^T \left( \beta(\omega, t) + \sqrt{T-t} x \right)} \right] = e^{\mu_n T + \sigma_n^T \left( \beta(\omega, t) + \frac{(T-t)}{2} \sigma_n \right)}.
\]
Relative Prices: Overview

\[
\frac{P_n(\omega, t)}{P_0(\omega, t)} = e^{\mu_n T + \sigma_n^T (\beta(\omega, t) + \frac{(T-t)}{2} \sigma_n)}
\]

\[
\mathbb{E}_x \left[ u' \left( W(\omega, t, x + \sqrt{T-t} \sigma_n) \right) \right] - \mathbb{E}_x \left[ u' \left( W(\omega, t, x) \right) \right]
\]

Given \(\beta(\omega, t)\), exchanging at time \(t\) one unit of the bond for one unit of the \(n\)th security results in an increase in expected wealth by its expected terminal-period dividend

\[
\mathbb{E}_x \left[ e^{\mu_n T + \sigma_n^T (\beta(\omega, t) + \sqrt{T-t} x)} \right] = e^{\mu_n T + \sigma_n^T (\beta(\omega, t) + \frac{(T-t)}{2} \sigma_n)}.
\]

\[\blacktriangleright\] In terms of terminal-period wealth, one unit of the \(n\)th security is equivalent to \(e^{\mu_n T + \sigma_n^T (\beta(\omega, t) + \frac{(T-t)}{2} \sigma_n)}\) units of the bond.
Relative Prices: Overview

\[
\frac{P_n(\omega, t)}{P_0(\omega, t)} = e^{\mu_nT + \sigma_n^T(\beta(\omega, t) + \frac{(T-t)}{2}\sigma_n)}
\]

\[
\mathbb{E}_x \left[ u' \left( W(\omega, t, x + \sqrt{T-t}\sigma_n) \right) \right] = \mathbb{E}_x \left[ u' \left( W(\omega, t, x) \right) \right] + \mathbb{E}_x \left[ u' \left( W(\omega, t, x) \right) \right].
\]

Given \( \beta(\omega, t) \), exchanging at time \( t \) one unit of the bond for one unit of the \( n \)th security results in an increase in expected wealth by its expected terminal-period dividend

\[
\mathbb{E}_x \left[ e^{\mu_nT + \sigma_n^T(\beta(\omega, t) + \sqrt{T-t}\sigma_n)} \right] = e^{\mu_nT + \sigma_n^T(\beta(\omega, t) + \frac{(T-t)}{2}\sigma_n)}.
\]

- In terms of terminal-period wealth, one unit of the \( n \)th security is equivalent to
  \[e^{\mu_nT + \sigma_n^T(\beta(\omega, t) + \frac{(T-t)}{2}\sigma_n)} \] units of the bond.

- In expected marginal utility terms, any future realization
  \( \sqrt{T-t}\sigma_n \) of the stochastic process \( \beta(\omega, T) - \beta(\omega, t) \)
  must be translated by \((T-t)\sigma_n \).
Dynamics of Absolute Prices

The Wealth Effect

\[
\frac{\partial p_0(\omega, t)}{\partial \beta_k(\omega, t)} = \mathbb{E}_x \left[ u''(W(\omega, t, x)) \frac{\partial W(\omega, t, x)}{\partial \beta_k(\omega, t)} \right]
\]
Dynamics of Absolute Prices

The Wealth Effect

\[
\frac{\partial p_0(\omega, t)}{\partial \beta_k(\omega, t)} = \mathbb{E}_x \left[ u''(W(\omega, t, x)) \frac{\partial W(\omega, t, x)}{\partial \beta_k(\omega, t)} \right]
\]

The Wealth, Own-dividend, and Asset-riskiness Effects

\[
\frac{\partial P_n(\omega, t)}{\partial \beta_k(\omega, t)} = e^{\mu_n T + \sigma_n^T \beta(\omega, t) + \frac{(T-t)}{2} \sigma_n \sigma_n^T} P_0(\omega, t)
\]

\[
+ \sigma_{nk} e^{\mu_n T + \sigma_n^T \beta(\omega, t) + \frac{(T-t)}{2} \sigma_n \sigma_n^T} P_0(\omega, t)
\]

\[
+ \frac{\partial}{\partial \beta_k(\omega, t)} \text{Cov}_x \left[ u'(W(\omega, t, x)), e^{\mu_n T + \sigma_n^T \beta(\omega, t) + \sqrt{T-t} x} \right]
\]
The Wealth Effect of $d\beta_k (\omega, t)$

$$\frac{\partial P_0 (\omega, t)}{\partial \beta_k (\omega, t)} = \mathbb{E}_x \left[ u'' (W (\omega, t, x)) \frac{\partial W (\omega, t, x)}{\partial \beta_k (\omega, t)} \right]$$
The Wealth Effect of $d\beta_k(\omega, t)$

$$\frac{\partial P_0(\omega, t)}{\partial \beta_k(\omega, t)} = \mathbb{E}_x \left[ u''(W(\omega, t, x)) \frac{\partial W(\omega, t, x)}{\partial \beta_k(\omega, t)} \right]$$

For any future realization $\sqrt{T-t}x \sim \mathcal{N}(0, (T-t)I_K)$ of the stochastic process $\beta(\omega, T) - \beta(\omega, t)$, $d\beta(\omega, t)$ corresponds to revealing information that changes the $(\mathcal{F}_t$-conditional) terminal-period dividend of every security $i \in \{1, \ldots, n\}$ by $\sigma_{ik}e^{\mu_kT+\sigma_i^T(\beta(\omega,t)+\sqrt{T-t}x)}$ and the terminal-period endowment by $\frac{\partial\rho(\beta(\omega,t)+\sqrt{T-t}x)}{\partial \beta_k(\omega,t)}$, we get the corresponding change in the terminal-period wealth.
The Wealth Effect of $d\beta_k(\omega, t)$

$$\frac{\partial P_0(\omega, t)}{\partial \beta_k(\omega, t)} = \mathbb{E}_x \left[ u''(W(\omega, t, x)) \frac{\partial W(\omega, t, x)}{\partial \beta_k(\omega, t)} \right]$$

For any future realization $\sqrt{T-t}x \sim \mathcal{N}(0, (T-t)I_K)$ of the stochastic process $\beta(\omega, T) - \beta(\omega, t)$, $d\beta(\omega, t)$ corresponds to revealing information that changes the $(\mathcal{F}_t$-conditional) terminal-period dividend of every security $i \in \{1, ..., n\}$ by $\sigma_{ik} \epsilon_i T + \sigma_i^T (\beta(\omega,t) + \sqrt{T-t}x)$ and the terminal-period endowment by $\frac{\partial \rho(\beta(\omega,t)+\sqrt{T-t}x)}{\partial \beta_k(\omega,t)}$, we get the corresponding change in the terminal-period wealth.

Other things being equal, due to the agent’s risk aversion ($u''(\cdot) < 0$), this induces a change in the opposite direction on the marginal utility of terminal-period wealth.
The Own-dividend Effect $\partial \beta_k(\omega, t)$

$$\sigma_{nk} e^{\mu_n T + \sigma_n^T (\beta(\omega,t) + \frac{(T-t)}{2} \sigma_n)} P_0(\omega, t)$$

The change in $\beta_k(\omega, t)$ also alters the ($\mathcal{F}_t$-conditional) expected terminal-period dividend of the $n$th security by

$$\sigma_{nk} e^{\mu_n T + \sigma_n^T (\beta(\omega,t) + \frac{(T-t)}{2} \sigma_n)}.$$
The Own-dividend Effect $\partial \beta_k (\omega, t)$

$$\sigma_{nk} e^{\mu_n T + \sigma_n^T (\beta(\omega,t) + \frac{(T-t)}{2} \sigma_n)} P_0 (\omega, t)$$

The change in $\beta_k (\omega, t)$ also alters the ($\mathcal{F}_t$-conditional) expected terminal-period dividend of the $n$th security by

$$\sigma_{nk} e^{\mu_n T + \sigma_n^T (\beta(\omega,t) + \frac{(T-t)}{2} \sigma_n)}.$$ 

Suppose that $\beta_k (\omega, t)$ increases. If $\sigma_{nk} > 0$ ($\sigma_{nk} < 0$), the expected terminal-period dividend will now be higher (lower). Due to non-satiation ($u' (\cdot) > 0$), the agent is now more (less) willing to hold the $n$th security.
The Own-dividend Effect $\partial \beta_k (\omega, t)$

$$\sigma_{nk} e^{\mu_n T + \sigma_n^T \left( \beta(\omega, t) + \frac{(T-t)}{2} \sigma_n \right)} P_0 (\omega, t)$$

The change in $\beta_k (\omega, t)$ also alters the ($\mathcal{F}_t$-conditional) expected terminal-period dividend of the $n$th security by

$$\sigma_{nk} e^{\mu_n T + \sigma_n^T \left( \beta(\omega, t) + \frac{(T-t)}{2} \sigma_n \right)}.$$

Suppose that $\beta_k (\omega, t)$ increases. If $\sigma_{nk} > 0$ ($\sigma_{nk} < 0$), the expected terminal-period dividend will now be higher (lower). Due to non-satiation ($u'(\cdot) > 0$), the agent is now more (less) willing to hold the $n$th security.

Since she must hold exactly its net supply in equilibrium, $P_n (\omega, t)$ must rise (fall) by

$$\sigma_{nk} e^{\mu_n T + \sigma_n^T \left( \beta(\omega, t) + \frac{(T-t)}{2} \sigma_n \right)} P_0 (\omega, t).$$
The Asset-riskness Effect of $\partial \beta_k (\omega, t)$

$$\frac{\partial}{\partial \beta_k (\omega, t)} \text{Cov}_x \left[ u' (W (\omega, t, x)), e^{\mu_n T + \sigma_n (\beta(\omega, t) + \sqrt{T-t}x)} \right]$$

For any future realization $\sqrt{T-t}x$, the extent to which $d\beta_k (\omega, t)$ alters the equilibrium price $P_n (\omega, t)$, through a change in the marginal utility of terminal-period wealth $u' (W (\omega, t, x))$, depends on the realization of the terminal-period dividend $e^{\mu_n T + \sigma_n T (\beta(\omega, t) + \sqrt{T-t}x)}$. Similarly, the extent to which $d\beta_k (\omega, t)$ alters $P_n (\omega, t)$, through a change in the $n$th terminal-period dividend, depends on the realization of the marginal utility of terminal-period wealth. In other words, $d\beta_k (\omega, t)$ affects $P_n (\omega, t)$ also via changes in the correlation of the marginal utility of terminal-period wealth with the terminal-period dividend of the $n$th risky security.
The Asset-riskness Effect: Intuition

Let \( \frac{\partial W(\cdot)}{\partial \beta_k(\omega, t)} > 0, \sigma_{nk} = 0. \) Due to RA, \( u'(W(\omega, t, x)) \) falls.
The Asset-riskness Effect: Intuition

\[
\frac{\partial}{\partial \beta_k (\omega, t)} \text{Cov}_x \left[ u' (W (\omega, t, x)) , e^{\mu_n T + \sigma_n (\beta(\omega, t) + \sqrt{T-t} x)} \right]
\]

Let \( \frac{\partial W(\cdot)}{\partial \beta_k (\omega, t)} > 0, \sigma_{nk} = 0 \). Due to RA, \( u' (W (\omega, t, x)) \) falls.

Under non-increasing absolute risk-aversion, the decrease in \( u' (W (\omega, t, x)) \) is smaller when the \( n \)th terminal-period dividend is large and larger when the dividend is small.

e.g. CARA: \( u'' (W (\omega, t, x)) = -r_A u' (W (\omega, t, x)) \)
The Asset-riskness Effect: Intuition

\[
\frac{\partial}{\partial \beta_k (\omega, t)} \text{Cov}_x \left[ u' (W (\omega, t, x)), e^{\mu_n T + \sigma_n (\beta(\omega, t) + \sqrt{T-t}x)} \right]
\]

Let \( \frac{\partial W(\cdot)}{\partial \beta_k (\omega, t)} > 0, \sigma_{nk} = 0 \). Due to RA, \( u' (W (\omega, t, x)) \) falls.

Under non-increasing absolute risk-aversion, the decrease in \( u' (W (\omega, t, x)) \) is smaller when the \( n \)th terminal-period dividend is large and larger when the dividend is small.

E.g. CARA: \( u'' (W (\omega, t, x)) = -r_A u' (W (\omega, t, x)) \)

Hence, the marginal utility of terminal-period wealth and the terminal-period dividend of the security become less negatively correlated. The perceived “riskiness” of the security falls and this has a positive effect on its equilibrium price.
For $\sigma_{nk} \neq 0$, we get another component working in the opposite direction.

The covariance of the marginal utility of terminal-period wealth with the terminal-period dividend becomes

$$e^{\sigma_{nk} \beta_k(\omega, t)} \text{Cov}_x \left[ u' (W(\omega, t, x) + dW(\omega, t, x)) , D_n(\omega, T) \right]$$
The Asset-riskness Effect: Intuition (Cont.)

In the $\sigma_{nk} = 0$ case, I argued that $\text{Cov}_x [u' (W (\omega, t, x) + dW (\omega, t, x)), D_n (\omega, T)]$ is less negative than the initial covariance of the marginal utility of terminal-period wealth with the terminal-period dividend, $\text{Cov}_x \left[ u' (W (\omega, t, x)), e^{\mu_n T + \sigma_n^\top (\beta(\omega, t) + \sqrt{T-t}x)} \right]$. 
The Asset-riskness Effect: Intuition (Cont.)

In the $\sigma_{nk} = 0$ case, I argued that
\[ \text{Cov}_x \left[ u'(W(\omega, t, x) + dW(\omega, t, x)), D_n(\omega, T) \right] \]
is less negative than the initial covariance of the marginal utility of terminal-period wealth with the terminal-period dividend,
\[ \text{Cov}_x \left[ u'(W(\omega, t, x)), e^{\mu_n T + \sigma_n^T (\beta(\omega, t) + \sqrt{T-t} x)} \right]. \]

Let, for example, $\sigma_{nk} > 0$ and $d\beta_k(\omega, t) > 0$. The $n$th terminal-period dividend may increase now sufficiently to make
\[ e^{\sigma_{nk} d\beta_k(\omega, t)} \text{Cov}_x \left[ u'(W(\omega, t, x) + dW(\omega, t, x)), D_n(\omega, T) \right] \]
more negative than \[ \text{Cov}_x \left[ u'(W(\omega, t, x)), D_n(\omega, T) \right]. \]
The Asset-riskness Effect: Intuition (Cont.)

In the $\sigma_{nk} = 0$ case, I argued that
\[ \text{Cov}_x [u'(W(\omega, t, x) + dW(\omega, t, x)), D_n(\omega, T)] \]
is less negative than the initial covariance of the marginal utility of terminal-period wealth with the terminal-period dividend,
\[ \text{Cov}_x \left[ u'(W(\omega, t, x)), e^{\mu_n T + \sigma_n^T(\beta(\omega, t) + \sqrt{T-t}x)} \right]. \]

Let, for example, $\sigma_{nk} > 0$ and $d\beta_k(\omega, t) > 0$. The $n$th terminal-period dividend may increase now sufficiently to make
\[ e^{\sigma_{nk} d\beta_k(\omega, t)} \text{Cov}_x [u'(W(\omega, t, x) + dW(\omega, t, x)), D_n(\omega, T)] \]
more negative than $\text{Cov}_x [u'(W(\omega, t, x)), D_n(\omega, T)]$.

The “riskiness” of the $n$th security may now be higher.
The Dynamics of Relative Prices

- The own-dividend effect pushes $P_{Dn}(\omega, t)$ in the direction of the change in $D_n(\omega, T)$.
The Dynamics of Relative Prices

- The own-dividend effect pushes \( P_{Dn}(\omega, t) \) in the direction of the change in \( D_n(\omega, T) \).

- Under DARA, the combination of the wealth effects on \( P_n(\omega, t) \) and \( P_0(\omega, t) \) pulls the relative price \( \frac{P_n(\omega, t)}{P_0(\omega, t)} \) in the direction of changes in terminal-period wealth.
The Dynamics of Relative Prices

- The own-dividend effect pushes $P_{D_n}(\omega, t)$ in the direction of the change in $D_n(\omega, T)$.

- Under DARA, the combination of the wealth effects on $P_n(\omega, t)$ and $P_0(\omega, t)$ pulls the relative price $\frac{P_n(\omega, t)}{P_0(\omega, t)}$ in the direction of changes in terminal-period wealth.

- The asset-riskiness effect may push $P_n(\omega, t)$ in the opposite direction than the above two.
Theorem 2.1

For any $n \in \{1, \ldots, N\}$,

\[
\sum_{k=1}^{K} \sigma_{nk} \frac{\partial}{\partial \beta_k (\omega, t)} \left( \frac{P_n (\omega, t)}{P_0 (\omega, t)} \right) \geq 0
\]

with the inequality being strict unless $\sigma_n = 0 \in \mathbb{R}^K$. 
Theorem 2.1

For any $n \in \{1, \ldots, N\}$,

$$\sum_{k=1}^{K} \sigma_{nk} \frac{\partial}{\partial \beta_k(\omega, t)} \left( \frac{P_n(\omega, t)}{P_0(\omega, t)} \right) \geq 0$$

with the inequality being strict unless $\sigma_n = 0 \in \mathbb{R}^K$.

Immediate Corollaries:

- $K = 1$: $\sigma_n \frac{\partial}{\partial \beta_k(\omega, t)} \left( \frac{P_n(\omega, t)}{P_0(\omega, t)} \right) \geq 0 \ \forall n$.

- $K > 1$ and $D_n(\omega, T)$ depends only on one source of uncertainty, ($\sigma_n^T = \sigma_{nm}e_m$): $\sigma_{nm} \frac{\partial}{\partial \beta_m(\omega, t)} \left( \frac{P_n(\omega, t)}{P_0(\omega, t)} \right) \geq 0$.

- $N = K > 1$ and $\Sigma$ is diagonal, $\Sigma = [\sigma_{kk}e_k]_{k \in \{1, \ldots, K\}}$: $\sigma_{kk} \frac{\partial}{\partial \beta_k(\omega, t)} \left( \frac{P_k(\omega, t)}{P_0(\omega, t)} \right) \geq 0 \ \forall n$. 
Theorem 2.1: Intuition

Let

(i) The terminal-period dividend of the $n$th security vary with only one source of uncertainty, say $\beta_m(\omega, t)$:

$$D_n(\omega, T) = e^{\mu_n T + \sigma_{nm}(\beta_m(\omega,t)+\sqrt{T-t}x_m)}$$

(ii) $\beta_m(\omega, t)$ affect the terminal-period wealth only through the $n$th dividend:

$$\frac{\partial \rho(\cdot)}{\partial \beta_m(\omega, T)} = 0 \quad \forall \omega \in \Omega \text{ and }$$

$$\sigma_{n'm} = 0 \quad \forall n' \in \{1, ..., N\} \setminus \{n\}.$$
Theorem 2.1: Intuition

Let

(i) The terminal-period dividend of the $n$th security vary with only one source of uncertainty, say $\beta_m(\omega, t)$:

$$D_n(\omega, T) = e^{\mu_n T + \sigma_{nm}(\beta_m(\omega, t) + \sqrt{T-t} \cdot m)}$$

(ii) $\beta_m(\omega, t)$ affect the terminal-period wealth only through the $n$th dividend:

$$\frac{\partial \rho(\cdot)}{\partial \beta_m(\omega, T)} = 0 \ \forall \omega \in \Omega \quad \text{and} \quad \sigma_{n'm} = 0 \ \forall n' \in \{1, \ldots, N\} \setminus \{n\}.$$ 

Let the $m$th component of the Brownian process change from $\beta_m(\omega, t)$ to $\beta_m(\omega, t) + d\beta_m(\omega, t)$. The terminal-period wealth changes only through the $n$th terminal-period dividend by

$$e^{\sigma_{nm} d\beta_m(\omega, t)} A_n(\omega, T)$$
Theorem 2.1: Intuition (Cont.)

Since the agent is everywhere non-satiated ($u'(\cdot) > 0$) and the terminal-period endowment is unaffected by the change $d\beta_m(\omega, t)$, her preferences for the risky asset change in the direction of First Degree Stochastic Dominance (FSD).
Theorem 2.1: Intuition (Cont.)

Since the agent is everywhere non-satiated \((u'(\cdot) > 0)\) and the terminal-period endowment is unaffected by the change \(d\beta_m(\omega, t)\), her preferences for the risky asset change in the direction of First Degree Stochastic Dominance (FSD).

Suppose that \(\beta_m(\omega, t)\) increases (decreases).
For \(\sigma_{nm} > 0\), the new terminal-period dividend dominates (is dominated by) the old in the sense of FSD. The agent is now more (less) willing to hold the stock. Moreover, the wealth effect on the equilibrium price of the bond is negative (positive).
Theorem 2.1: Intuition (Cont.)

Since the agent is everywhere non-satiated \( (u'(\cdot) > 0) \) and the terminal-period endowment is unaffected by the change \( d\beta_m(\omega, t) \), her preferences for the risky asset change in the direction of First Degree Stochastic Dominance (FSD).

Suppose that \( \beta_m(\omega, t) \) increases (decreases). For \( \sigma_{nm} > 0 \), the new terminal-period dividend dominates (is dominated by) the old in the sense of FSD. The agent is now more (less) willing to hold the stock. Moreover, the wealth effect on the equilibrium price of the bond is negative (positive).

For \( \sigma_{nm} < 0 \), the old terminal-period dividend dominates (is dominated by) the new in terms of FSD, whereas the wealth effect on \( p_B(\omega, t) \) is positive (negative).

In either case, \( \sigma_{nm} \frac{\partial}{\partial \beta_m(\omega, t)} \left( \frac{P_n(\omega, t)}{P_0(\omega, t)} \right) > 0. \)
Cross-correlations ($K > 1$)

Let $\sigma_{nk} = 0$. The implications of Theorem 2.1 for

$$\frac{\partial}{\partial \beta_k(\omega,t)} \left( \frac{P_n(\omega,t)}{P_0(\omega,t)} \right)$$

are quite subtle, even when the model has only one security ($N = 1$).

It is instructive to begin by considering settings where the terminal-period dividend of the $n$th security depends on the realization of only one source of uncertainty:

$$D_n(\omega, T) = e^{\mu_n T + \sigma_{nm}(\beta_m(\omega,t) + \sqrt{T-t}x_m)}.$$
CARA Utility: Proposition 2.1

For \( n \in \{1, \ldots, N\} \), suppose that

(i) \( D_n(\omega, t) = e^{\mu_n T + \sigma_{nm}(\beta_m(\omega, t) + \sqrt{T-t}x_m)} \)

(ii) The terminal-period wealth process is given by

\[
W(\omega, t, x) = W_{-k}(\omega, t, x_{-k}) + W_k(\omega, t, x_k)
\]

(ii) The terminal-period utility \( u(\cdot) \) exhibits CARA.

The relative price process \( \frac{P_n(\omega, t)}{P_0(\omega, t)} \) is independent of the \( k \)th source of uncertainty, for \( k \neq m \):

\[
\frac{\partial}{\partial \beta_k(\omega, t)} \left( \frac{P_n(\omega, t)}{P_0(\omega, t)} \right) = 0
\]
CARA Utility: Proposition 2.2

For \( n \in \{1, \ldots, N\} \), suppose that

(i) \( D_n (\omega, t) = e^{\mu_n T + \sigma_{nm} (\beta_m (\omega, t) + \sqrt{T-t} x_m)} \)

(ii) The terminal-period wealth process is given by

\[
W (\omega, t, x) = W_{-m} (\omega, t, x_{-m}) + W_m (\omega, t, x_m)
\]

(iii) The terminal-period utility \( u (\cdot) \) exhibits CARA.

The relative price process \( \frac{P_n (\omega, t)}{P_0 (\omega, t)} \) is independent of the \( k \)th source of uncertainty, for \( k \neq m \):

\[
\frac{\partial}{\partial \beta_k (\omega, t)} \left( \frac{P_n (\omega, t)}{P_0 (\omega, t)} \right) = 0
\]
Dynamic Completeness with CARA

Suppose that:

- The matrix $\Sigma$ of factor loadings is diagonal:
  $N = K$ and $\sigma = [\sigma_{kk}e_k]_{k \in \{1, \ldots, K\}}, \sigma_{kk} \in \mathbb{R}$.
- The terminal-period endowment process is given by:

$$\rho(\beta(\omega, T)) = \sum_{k=1}^{K} \rho_i \left( \beta_k(\omega, t) + \sqrt{T - tx_k} \right)$$

for some continuous functions $\rho_k : \mathbb{R} \rightarrow \mathbb{R}_+$.

The terminal-period wealth can now be written as

$$W(\omega, t, x) = \sum_{k=1}^{K} W_k(\omega, t, x_k)$$
By Theorem 2.1 and either of Propositions 2.1 or 2.2,

\[ \sigma_{nn} \frac{\partial}{\partial \beta_k} \left( \frac{P_n(\omega, t)}{P_0(\omega, t)} \right) > 0 \quad \text{if } k = n \]

\[ \frac{\partial}{\partial \beta_k} \left( \frac{P_n(\omega, t)}{P_0(\omega, t)} \right) = 0 \quad \text{if } k \neq n \]

The dispersion matrix of the equilibrium relative prices

\[ J(\beta(\omega, t)) = \left[ \frac{\partial}{\partial \beta_k(\omega, t)} \left( \frac{P_n(\omega, t)}{P_0(\omega, t)} \right) \right]_{n,k \in \{1, \ldots, K\}} \]

is diagonal.
“Contagion” with CARA: Example

Let the terminal-period utility function \( u(\cdot) \) exhibit CARA. Suppose also that

(i) \( D_n(\omega, T) \) depends only on the \( m \)th component of the Brownian process \( \beta(\omega, T) \) for some \( m \in \{1, \ldots, K\} \): 
\[
\sigma_n = \sigma_{nm}e_m.
\]

(ii) For \( k \neq m \), \( \beta_k(\omega, T) \) does not affect any component of terminal-period wealth other than the \( n' \)th \( (n' \neq n) \) terminal-period dividend.

(iii) \( A_{n'}(\omega, T) \) depends only on the \( k \)th and \( m \)th components of \( \beta(\omega, T) \): 
\[
\sigma_{n'} = \sigma_{n'm}e_m + \sigma_{n'k}e_k.
\]

Then, \( \frac{P_n(\omega,t)}{P_0(\omega,t)} \) is monotone in \( \beta_k(\omega, t) \):

\[
\sigma_{n'k} \frac{\partial}{\partial \beta_k(\omega, t)} \left( \frac{P_n(\omega, t)}{P_0(\omega, t)} \right) \leq 0 \quad \text{with equality iff} \quad \sigma_{n'k} = 0
\]
Any matrix $\Sigma$ satisfying these conditions is non-diagonal.

\[
\begin{pmatrix}
\sigma_{11} & 0 \\
\sigma_{21} & \sigma_{22}
\end{pmatrix}
\]

\[
\sigma_{11} \frac{\partial}{\partial \beta_1(\omega,t)} \begin{pmatrix} P_1(\omega,t) \\ P_0(\omega,t) \end{pmatrix} > 0
\]

\[
\sigma_{22} \frac{\partial}{\partial \beta_2(\omega,t)} \begin{pmatrix} \rho_{A_1}(\omega,t) \\ \rho_{B}(\omega,t) \end{pmatrix} < 0
\]
“Contagion” with DARA: Example

Let the terminal-period utility function $u(\cdot)$ exhibit DARA. Suppose also that

(i) The matrix $\Sigma$ of factor loadings is diagonal: $N = K$ and $\sigma = [\sigma_{kk}e_k]_{k \in \{1, \ldots, K\}}$, $\sigma_{kk} \in \mathbb{R}$.

(ii) The terminal-period endowment process is deterministic:

$$W(\omega, t, x) = \rho + \sum_{n'=1}^{N} e^{\mu_n T + \sigma_{nn}(\beta_n(\omega, t) + \sqrt{T-t}x_n)}$$

Then,

$$\sigma_{nn} \frac{\partial}{\partial \beta_n(\omega, t)} \left( \frac{P_n(\omega, t)}{P_0(\omega, t)} \right) \geq 0 \quad \text{with equality iff} \quad \sigma_{n'n} = 0$$
"Contagion" with DARA: Example

\[ \Sigma = \begin{pmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{pmatrix} \]

By Theorem 2.1 and the preceding result

\[ \sigma_{11} \frac{\partial}{\partial \beta_1 (\omega, t)} \left( \frac{P_1 (\omega, t)}{P_0 (\omega, t)} \right) > 0 \]
\[ \sigma_{11} \frac{\partial}{\partial \beta_1 (\omega, t)} \left( \frac{P_2 (\omega, t)}{P_0 (\omega, t)} \right) > 0 \]
\[ \sigma_{22} \frac{\partial}{\partial \beta_2 (\omega, t)} \left( \frac{P_1 (\omega, t)}{P_0 (\omega, t)} \right) > 0 \]
\[ \sigma_{22} \frac{\partial}{\partial \beta_2 (\omega, t)} \left( \frac{P_2 (\omega, t)}{P_0 (\omega, t)} \right) > 0 \]
Dynamic Completeness with DARA: Example

$$\sigma = \begin{pmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{pmatrix}$$

By Theorem 2.1 and Corollary 2.3,

$$\text{sign} \{ J(\beta(\omega, t))_{nk} \} = \text{sign} \{ \sigma_{kk} \} \quad n, k \in \{1, 2\}$$

$$|J(\beta(\omega, t)) + J(\beta(\omega, t))^T| = 4J(\beta(\omega, t))_{11} J(\beta(\omega, t))_{22} - (J(\beta(\omega, t))_{12} + J(\beta(\omega, t))_{21})^2$$

If $\sigma_{11}\sigma_{22} < 0$,

$$|J(\beta(\omega, t)) + J(\beta(\omega, t))^T| < 0 \quad \forall (\omega, t) \in \Omega \times [0, T].$$
Multiple Sources of Uncertainty in $D_n(\omega, T)$

The results generalize when the terminal-period payoff of the $n$th asset depends on multiple sources of uncertainty but $\sigma_{nk}$ remains zero.
Example: Dynamic Incompleteness with CRRA

Let the terminal-period utility exhibit CRRA:

\[ u(c) = \gamma c^\alpha, \alpha, \gamma < 0 \quad \text{or} \quad u(c) = \ln c \]

Suppose also that

(i) There is no terminal-period endowment: \( \rho(\cdot) = 0 \), and

(ii) \( \sigma_{n'}^T \sigma_n = s_n \quad \forall n' \in \{1, ..., N\} \).

Then, \( \frac{P_n(\omega,t)}{P_0(\omega,t)} \) is monotone in \( \beta_k(\omega, t) \):

\[
\frac{\partial}{\partial \beta_k} \left( \frac{P_n}{P_0} \right) = \sigma_{nk} \left( \frac{\mathbb{E}_x [u'(W(x))]}{P_0} \right)^2 e^{\mu_n T + \sigma_n^T \beta} + \frac{s_n (2\alpha - 1)(T-t)}{2}
\]
Example: Dynamic Incompleteness with CRRA (Cont.)

The $n$th row of the Jacobian of equilibrium relative prices

$$ j(\omega, t)_n = \left( \frac{\mathbb{E}_y [u'(W(\omega, t, y))]}{P_0(\omega, t)} \right)^2 \text{e}^{\mu_n T + \sigma_n^T \beta(\omega, t) + \frac{sn(2\alpha-1)(T-t)}{2} \sigma_n^T} \right) $$

If $N = K$ and (ii) holds for every security $n \in \{1, \ldots, N\}$, the determinant of the Jacobian matrix will be

$$ |J(\omega, t)| = \left( \frac{\mathbb{E}_y [u'(W(\omega, t, y))]}{p_B(\omega, t)} \right)^2 \left( \prod_{n=1}^{K} \text{e}^{\mu_n T + \sigma_n \beta(\omega, t) + \frac{(2\alpha-1)(T-t)sn}{2}} \right) |\Sigma| $$

Dynamic incompleteness as $\Sigma$ is singular.
Example: Dynamic Incompleteness with CRRA (Cont.)

\[
\Sigma = \begin{pmatrix}
\sigma_{11} & \sigma_{12} \\
\sigma_{21} & \sigma_{22}
\end{pmatrix} \quad \frac{\sigma_{11}}{\sigma_{12}} = \frac{\sigma_{11} - \sigma_{22}}{\sigma_{12} - \sigma_{21}}
\]

\[
\sigma_{11} \frac{\partial}{\partial \beta_1(\omega, t)} \left( \frac{P_1(\omega, t)}{P_0(\omega, t)} \right) > 0
\]

\[
\sigma_{12} \frac{\partial}{\partial \beta_2(\omega, t)} \left( \frac{P_1(\omega, t)}{P_0(\omega, t)} \right) > 0
\]

Dynamic incompleteness as \(\Sigma\) is singular.