Long Term Risk: An operator approach

With Lars Peter Hansen

- Long run risk return tradeoff
- Motivation
  - Evaluation of economic models of preferences and technologies using asset prices.
    - Market microstructure, transaction costs... may make it hard to evaluate these models using short run data.
    - Behavioral biases
  - How risk averse agents value the risks in permanent shocks
  - Long run risk-return frontier
  - Complementary to work using short run data (Bansal-Yaron...)
  - Hansen, Heaton and Li
- **Markovian structure**

Sustainable development

- Risks for which markets that may reveal information are absent
- Risks for which markets are present
  - Technological vs. economic risk.
  - Correlation with other factors that determine "utility."
  - Catastrophe insurance, pricing of CAT bonds.

Stochastic discount factor

- X_t a Markov process, $\mathcal{F}_t$ the associated (completed) filtration.
- A *Stochastic Discount Factor* S is a strictly positive adapted process such that if $s \le t$

$$\frac{E\left[S_t \Pi_t | \mathcal{F}_s\right]}{S_s} \tag{1}$$

  is the price at time s of a claim to the payoff Π_t at t.

- $$\mathbb{S}_t \psi(x) = E\left[S_t \psi(X_t) | X_0 = x\right],$$

  is the time-zero price of payoff $\psi(X_t)$.

- Law of one price
- Garman (1984), Rogers (1998)
- $\mathbb{S}_0 = \mathbb{I}$ and $\mathbb{S}_{t+u} = \mathbb{S}_t \mathbb{S}_u$
- Breeden model

- $dX_t = \vartheta(\bar{x} - X_t) + \sigma dB_t$
- Per-capita consumption

$$dc_t = X_t dt + \gamma dB_t.$$

where $c_t = \log(C_t)$

- Representative investor preferences are given by:

$$E \int_0^\infty \exp(-\mathsf{b}t) \frac{{C_t}^{1-\mathsf{a}} - 1}{1 - \mathsf{a}}$$

for a and b strictly positive.

- The implied stochastic discount factor is $S_t = \exp(A_t^s)$ where

$$A_t^s = -\mathsf{a} \int_0^t X_s ds - \mathsf{b}t - \mathsf{a} \int_0^t \gamma dB_s.$$

- θ_t the *shift operator*:

$$(\theta_t X)_u = X_{t+u}.$$

- Since S_u only depends on the history of the Markov process X between dates 0 and u, $S_u(\theta_t)$ only depends on the history of X between dates t and $t+u$.

- Consider payoffs at $t+u$ that are indicator functions of sets of histories observable at $t+u$, i.e. sets $B \in \mathcal{F}_{t+u}$, and again using intermediate trading dates and the law of one price one obtains:

$$E[S_{t+u}\mathbf{1}_B|X_0] = E[S_t E[S_u(\theta_t)\mathbf{1}_B|\mathcal{F}_t]|X_0] = E[S_t S_u(\theta_t)\mathbf{1}_B|X_0]$$

- $S_0 = 1$ and $S_{t+u} = S_t S_u(\theta_t)$.

- S_t is a **multiplicative functional.**

Generalizations

- G a growth process
- G adapted, $G_0 = 1$ and $G_{t+u} = G_t G_u(\theta_t)$
- $M = SG$ is also multiplicative
- $$\mathbb{M}_t \psi(x) = E\left[M_t \psi(X_t) | X_0 = x\right],$$

  is the time-zero price of payoff $D_0 G \psi(X_t)$.
- Valuation functional V
  - VS a martingale
  - $$\mathbb{M}_t \mathbf{1} = E\left[V_t | X_0 = x\right],$$

Objective: Multiplicative decomposition

- Establish decomposition for multiplicative functionals:

$$M_t = \exp(\rho t)\hat{M}_t \left[\frac{\varphi(X_0)}{\varphi(X_t)}\right]$$

  where

  - ρ is a deterministic growth rate;
  - $\hat{M}_t$ is a multiplicative martingale;
  - φ is a strictly positive function of the Markov state;

- If X is stationary, $\frac{\varphi(X_0)}{\varphi(X_t)}$ stationary component, $\hat{M}$ the martingale component of M, and ρ its growth rate.
  - Not entirely correct because of possible correlation between stationary and martingale components.

Implications of multiplicative decomposition

- If $\hat{M}$ is a martingale for $F \in \mathcal{F}_t$

$$\hat{P}r(F) = E[\hat{M}_t \mathbf{1}_F]$$

- X remains Markovian.

- $$E\left[M_t \psi(X_t) | X_0 = x\right] = \exp(\rho t)\phi(x)\hat{E}\left[\frac{\psi(X_t)}{\phi(X_t)} | X_0 = x\right]$$

- $\exp(-\rho t)\phi(X_t)$ as a *numeraire*. Applicable when the multiplicative process does not define a price.

- If X stationary under $\hat{P}r$ then in fact ρ is the asymptotic growth rate $\mathbb{M}_t$.

- If in addition to stationarity, recurrence holds

$$\lim_{t\to\infty} \hat{E}\left[\psi(X_t)|X_0 = x\right] = \hat{E}\left[\psi(X_t)\right].$$

$$\begin{aligned}\lim_{t\to\infty} e^{-\rho t} E[M_t \psi(X_t)|X_0 &= x] \\ &= \lim_{t\to\infty} E\left(\hat{M}_t \left[\frac{\psi(X_t)}{\phi(X_t)}\right]|X_0 = x\right)\phi(x) \\ &= \hat{E}\left[\frac{\psi(X_t)}{\phi(X_t)}\right]\phi(x)\end{aligned}$$

  - ρ is the (deterministic) growth rate
  - All state dependence is given by the eigenfunction ϕ

Long term bonds

- S a stochastic discount factor

$$S_t = \exp(\rho t)\hat{M}_t \frac{\phi(X_0)}{\phi(X_t)}$$

- Prices of long term discount bonds:

$$\exp(-\rho t)E\left(S_t|X_0 = x\right) \approx \hat{E}\left[\frac{1}{\phi(X_t)}\right]\phi(x)$$

- Alvarez and Jerman [2005] estimate the volatility of $\frac{\hat{M}_{t+1}}{\hat{M}_t}$ as a proportion of volatility of $\frac{S_{t+1}}{S_t}$. (around 75-100%.). Transitory component has low estimated conditional volatility.

Remainder of lecture

- Strategy to establish decomposition
  - Perron-Frobenius
- Long-run dominance
- Uniqueness
- Existence
- Example

Restrictions on Markov Process

- $\{X_t : t \geq 0\}$ be a continuous time Markov process on a state space D_0. The sample paths of $\{X_t : t \geq 0\}$ are continuous from the right and with left limits and we will sometimes also assume that this process is stationary and ergodic. Let $\mathcal{F}_t$ be completion of the sigma algebra generated by $\{X_u : 0 \leq u \leq t\}$.
- Semimartingale
- $X = X^c + X^j$
- X^j with a finite number of jumps in any finite interval and compensator $\eta[dy|x]dt$.
- An $\mathcal{F}_t$ n-dimensional Brownian motion $\{B_t\}$
- $\Sigma = \sigma\sigma'$
- $X_t^c = X_0 + \int_0^t \xi(X_u)du + \int_0^t \sigma(X_u)dB_u$
- (ξ, Σ, η)

Example

- $X = (X^v, X^m)$
-

$$
\begin{aligned}
dX^v_t &= \vartheta_v(\bar{x}_v - X^v_t) + \sqrt{X^v_t}\sigma_v dB^v_t, \\
dX^m_t &= \vartheta_m(\bar{x}_m - X^m_t) + \sigma_m dB^m_t
\end{aligned}
$$

  with $\vartheta_i > 0$, $\bar{x}^i > 0$ for $i = v, m$ and $2\kappa_v \bar{x}^v \geq |\sigma_v|^2$ and $B = \begin{bmatrix} B^v \\ B^m \end{bmatrix}$ is a bivariate standard Brownian motion.

- The parameter restrictions guarantee that there is a stationary distribution associated with X with support contained in $\mathbb{R}_+ \times \mathbb{R}$.

Parameterizing multiplicative functionals

- A real-valued process $\{M_t : t \geq 0\}$ adapted, right continuous with left limits. (**A functional**)
- The functional $\{M_t : t \geq 0\}$ is **multiplicative** if $M_0 = 1$, and $M_{t+u} = M_u(\theta_t)M_t$. Here θ is the shift operator.
- If M strictly positive $\log(M)$ will satisfy an additive property.
- A functional is **additive** if $A_0 = 0$ and $A_{t+u} = A_u(\theta_t) + A_t$, for each nonnegative t and u.
- Parameterize $\log(M)$

- (β, γ, κ) that satisfies:

  a) $\beta : D_0 \to \mathbb{R}$ and $\int_0^t \beta(X_u)du < \infty$ for every positive t;

  b) $\gamma : D_0 \to \mathbb{R}^m$ and $\int_0^t |\gamma(X_u)|^2 du < \infty$ for every positive t;

  c) $\kappa : D_0 \times D_0 \to \mathbb{R}$, $\kappa(x, x) = 0$ for all $x \in D_0$.

$$A_t = \int_0^t \beta(X_u)du + \int_0^t \gamma(X_u) \cdot dB_u + \sum_{0 \leq u \leq t} \kappa(X_u, X_{u-})$$

- $A_t = \psi(X_t) - \psi(X_0)$
- Exponential of additive processes (strictly positive multiplicative functionals).
  - Parameterized by the additive process (β, γ, κ)
- Product of multiplicative processes is multiplicative.

Example continued : SDF - Breeden model

- Per-capita consumption

$$dc_t = X_t^m dt + \sqrt{X_t^v}\gamma_v dB_t^v + \gamma_m dB_t^m.$$

where $c_t = \log(C_t)$

- Representative investor preferences are given by:

$$E\int_0^\infty \exp(-\mathsf{b}t)\frac{{C_t}^{1-\mathsf{a}} - 1}{1-\mathsf{a}}$$

for a and b strictly positive.

- The implied stochastic discount factor is $S_t = \exp(A_t^s)$ where

$$A_t^s = -\mathsf{a}\int_0^t X_s^m ds - \mathsf{b}t - \mathsf{a}\int_0^t \sqrt{X_s^v}\gamma_v dB_s^v - \mathsf{a}\int_0^t \gamma_m dB_s^m.$$

- local risk prices

$$\mathsf{a}(\sqrt{X_t^v}\gamma_v, \gamma_m).$$

SDF: Kreps-Porteous model

- Preferences satisfy recursion ($\mathsf{a} > 1$ and $\mathsf{b} > 0$):

$$\lim_{\epsilon \downarrow 0} \frac{E\left(W_{t+\epsilon} - W_t | \mathcal{F}_t\right)}{\epsilon} = W_t \left[\mathsf{b}(\mathsf{a}-1)c_t + \mathsf{b} \log W_t\right]$$

where $-W_t$ is the continuation value for the consumption plan.

- Bansal and Yarom (2004)
- Guess and verify:

$$W_t = \exp\left[(1-\mathsf{a})(\mathsf{w}_f X_t^f + \mathsf{w}_o X_t^o + c_t + \bar{\mathsf{w}})\right]$$

- $S_t = \exp(A_t^s) \times \exp(A_t^w)$
-
$$A_t^s = -\int_0^t X_s^m ds - \mathsf{b}t - \int_0^t \sqrt{X_s^v}\vartheta_v dB_s^v - \int_0^t \vartheta_m dB_s^m.$$
-
$$\begin{aligned} A_t^w &= (1-\mathsf{a})\int_0^t \sqrt{X_s^v}(\vartheta_v + \mathsf{w}_v\sigma_v)dB_s^v \\ &\quad +(1-\mathsf{a})\int_0^t (\vartheta_m + \mathsf{w}_m\sigma_m)dB_s^m \\ &\quad -\frac{(1-\mathsf{a})^2}{2}\int_0^t X_s^v \frac{|\vartheta_v + \mathsf{w}_v\sigma_v|^2}{2}ds - \frac{(1-\mathsf{a})^2}{2}t \end{aligned}$$

Generators

- Associate to each ψ a function χ such that $M_t\chi(X_t)$ is the "expected time derivative" of $M_t\psi(X_t)$.
- A Borel function ψ is in the domain of the **extended generator** $\mathbb{A}$ of the multiplicative functional M_t if for a Borel function χ, $N_t = M_t\psi(X_t) - \psi(X_0) - \int_0^t M_s\chi(X_s)ds$ is a local martingale with respect to the filtration $\{\mathcal{F}_t : t \geq 0\}$. The extended generator assigns χ to ψ and we write $\chi = \mathbb{A}\psi$.
- If ψ smooth and $M \equiv 1$ apply Ito's lemma.
- X described by $(\xi, \sigma\sigma', \eta)$, M_t the exponential of (β, γ, κ).
- $\mathbb{A}\psi(x) = [\beta(x) + \frac{|\gamma(x)|^2}{2} + \int (\exp[\kappa(y,x)] - 1)\, \eta(dy,x)]\psi(x) + [\xi(x) + \sigma(x)\gamma(x)]\frac{\partial\psi(x)}{\partial x} + \frac{1}{2}\mathrm{trace}(\sigma\sigma'\frac{\partial^2\psi(x)}{\partial x\partial x'}) + \int_{R^n - \{x\}}[\psi(y) - \psi(x)]\exp[\kappa(y,x)]\,\eta(dy|x)$ (Ito's Lemma)

Eigenfunctions and martingales

- A Borel function ϕ is an **eigenfunction** of the extended generator (with **eigenvalue** ρ) if $\mathbb{A}\phi = \rho\phi$.
- $N_t = M_t\phi(X_t) - \phi(X_0) - \rho\int_0^t M_s\phi(X_s)ds$ is a $\mathcal{F}_t$ local martingale.
- Set $Y_t = M_t\phi(X_t)$. Since $dN_t = dY_t - \rho Y_{t-}$, integration by parts yields:

$$\begin{aligned}\exp(-\rho t)Y_t - Y_0 &= -\int_0^t \rho\exp(-\rho s)Y_{s-}ds + \int_0^t \exp(-\rho s)dY_s \\ &= \int_0^t \exp(-\rho s)dN_s.\end{aligned}$$

- $\exp(-\rho t)M_t\phi(X_t)$ is a local martingale.

- A **principal eigenfunction** of the extended generator is an eigenfunction that is strictly positive.
- If ϕ is a principal eigenfunction,

$$M_t = \exp(\rho t)\hat{M}_t \left[\frac{\phi(X_0)}{\phi(X_t)}\right].$$

  where $\hat{M}_t = \exp(-\rho t) M_t \frac{\phi(X_t)}{\phi(X_0)}$ is a multiplicative local martingale.
- Will discuss assumptions under which $\hat{M}$ is actually a martingale.

Example: Risk-return frontier

- Cash flow process $D_0 G_t \psi(X_t)$
- $G_t = \exp(A^g_t)$ where

$$A^g_t = \delta t + \int_0^t \sqrt{X^v_s}\varpi_v dB^v_s + \int_0^t \varpi_m dB^m_s - \int_0^t \frac{X^v_s|\varpi_v|^2 + |\varpi_m|^2}{2} ds$$

  - $G = \exp(\delta t)\hat{G}$, $\hat{G}$ a martingale.
  - ϖ_v parameterizes B^v risk of cash flow, ϖ_m parameterizes B^m risk

- $M = SG = \exp(A) = \exp(A^s + A^g)$

$$A_t = (\delta - \mathsf{b})t \quad + \quad \int_0^t \sqrt{X^v_s}(-\mathsf{a}\gamma_v + \varpi_v) dB^v_s$$

$$+ \int_0^t (-\mathsf{a}\gamma_m + \varpi_m) dB^m_s \quad - \quad \int_0^t \left[\frac{X^v_s|\varpi_v|^2 + |\varpi_m|^2}{2} + \mathsf{a}X^m_s\right] ds$$

- Compute $\mathbb{A}$ (Ito's)
- Guess an eigenfunction of the form: $\exp(\mathsf{c}_v x^v + \mathsf{c}_m x^m)$.
- Real solutions if

$$H = [\vartheta_v + (\mathsf{a}\gamma_v - \varpi_v)\sigma_v]^2 - |\sigma_v|^2[|\mathsf{a}\gamma_v|^2 - |\varpi_v|^2] \geq 0$$

$$\begin{aligned} \mathsf{c}_m &= -\frac{\mathsf{a}}{\vartheta_m} \\ \mathsf{c}_v &= \frac{\vartheta_v + (\mathsf{a}\gamma_v - \varpi_v)\sigma_v \pm \sqrt{H}}{|\sigma_v|^2} \end{aligned}$$

Only the minus sign will lead to stationarity of X under the distorted probability distribution. Can check directly that $\hat{M}$ is a martingale and recurrence holds.

- The eigenvalue is
$\lambda = \delta - \mathsf{b} + \frac{|\mathsf{a}\gamma_m|^2}{2} - \mathsf{a}\gamma_m\varpi_m + \vartheta_v\bar{x}_v\mathsf{c}_v + \vartheta_m\bar{x}_m\mathsf{c}_m$
$+ (-\mathsf{a}\gamma_m + \varpi_m)\sigma_m\mathsf{c}_m + (\mathsf{c}_m)^2\frac{|\sigma_m|^2}{2}$

- $-\lambda$ is the decay rate in value of the cash flow over time.
- Risk adjusted asymptotic interest rate
  $-\lambda + \delta = \mathsf{b} - \frac{|\mathsf{a}\gamma_m|^2}{2} + \mathsf{a}\gamma_m \varpi_m - \vartheta_v \bar{x}_v \mathsf{c}_v - \vartheta_m \bar{x}_m \mathsf{c}_m$
  $- (-\mathsf{a}\gamma_m + \varpi_m)\sigma_m \mathsf{c}_m - (\mathsf{c}_m)^2 \frac{|\sigma_m|^2}{2}.$
- Risk-return frontier: mapping

$$(\varpi_v, \varpi_m) \hookrightarrow -\lambda + \delta$$

- The *long run risk prices* to the cash flow risk exposure to the B^m risk is:

$$a\gamma_m + \frac{a}{\vartheta_m}\sigma_m$$

- Cash flow risk exposure to B^v feeds through eigenfunction.

Long run dominance

Assumption 1. *The multiplicative functional M is strictly positive with probability one.*

Assumption 2. *There exists a probability measure $\hat{\varsigma}$ such that*

$$\int \hat{\mathbb{A}}\psi d\hat{\varsigma} = 0$$

for all ψ in the L^∞ domain of the generator $\hat{\mathbb{A}}$ of $\hat{M}$.

Assumption 3. *There exists a $\hat{\Delta} > 0$ such that the discretely sampled process $\{X_{\hat{\Delta}j} : j = 0, 1, ...\}$ is* **irreducible**. *That is, for any Borel set Λ of the state space $\mathcal{D}_0$ with $\hat{\varsigma}(\Lambda) > 0$,*

$$\hat{E}\left[\sum_{j=0}^{\infty} \mathbf{1}_{\{X_{\hat{\Delta}j} \in \Lambda\}} | X_0 = x\right] > 0$$

for all $x \in \mathcal{D}_0$.

Assumption 4. *The process X is* **Harris recurrent** *under the measure $\hat{P}r$. That is, for any Borel set Λ of the state space $\mathcal{D}_0$ with positive $\hat{\varsigma}$ measure,*

$$\hat{P}r\left\{\int_0^\infty \mathbf{1}_{\{X_t \in \Lambda\}} = \infty | X_0 = x\right\} = 1$$

for all $x \in \mathcal{D}_0$.

Among other things, this assumption guarantees that the stationary distribution $\hat{\varsigma}$ is unique.

- Sufficient conditions for Assumptions 1-4 using Liapunov functions (Meyn-Tweedie).
  - Restrictions on coefficients of M and X
  - X Feller
  - A function V is called **norm-like** if $\{x : V(x) \leq r\}$ is precompact for each $r > 0$.
  - Example: A sufficient condition for the existence of stationary distribution (Assumption 2) and for Harris recurrence (Assumption 4) is that there exists a norm-like function V for which

$$\frac{\mathbb{A}(\phi V)}{\phi} - \rho V = \hat{\mathbb{A}} V \leq -1$$

outside a compact subset of the state space.

Proposition 1. *Suppose that $\hat{M}$ is a martingale and satisfies Assumptions 1 - 4, and let $\Delta > 0$.*

a. For any ψ for which $\int (|\psi|/\phi) d\hat{\varsigma} < +\infty$

$$\lim_{j\to\infty} \exp(-\rho\Delta j)\mathbb{M}_{\Delta j}\psi = \phi \int \frac{\psi}{\phi} d\hat{\varsigma}$$

for almost all ($\hat{\varsigma}$) x.

b. For any ψ for which ψ/ϕ is bounded,

$$\lim_{t\to\infty} \exp(-\rho t)\mathbb{M}_t\psi = \phi \int \frac{\psi}{\phi} d\hat{\varsigma}$$

for $x \in \mathcal{D}_0$.

Uniqueness

- Under the above assumptions there exists at most one principal eigenfunction for which $\hat{M}$ is a martingale and X is stationary and recurrent under $\hat{P}r$.

- Proof: Consider two such principal eigenvectors, ϕ and ϕ^*, and let $\rho \geq \rho^*$ be the corresponding eigenvalues. Since $\hat{M}$ is a martingale, $\exp(\rho t)\phi(x) = \exp(\rho t)\phi(x)E[\hat{M}_t|X_0 = x] = E[M_t\phi(X_t)|X_0 = x]$, and since $\hat{M}^*$ is a martingale $\exp(\rho^* t)\phi^*(x) = E[M_t\phi^*(X_t)|X_0 = x]$. Thus
$$E\left[\hat{M}_t \frac{\phi^*(X_t)}{\phi(X_t)}|X_0 = x\right] = \exp[(\rho^* - \rho)t]\frac{\phi^*(x)}{\phi(x)}.$$

Since the discrete time sampled Markov process associated with $\hat{M}$ is stationary, aperiodic and Harris recurrent the left-hand side converges to:

$$\hat{E}\left[\frac{\phi^*(X_t)}{\phi(X_t)}\right].$$

for $t = \Delta j$ as $j \to \infty$. While the limit could be $+\infty$, it must be strictly positive, implying, since $\rho \geq \rho^*$, that $\rho^* = \rho$ and

$$\hat{E}\left[\frac{\phi^*(X_t)}{\phi(X_t)}\right] = \frac{\phi^*(x)}{\phi(x)}.$$

Hence the ratio of the two eigenfunctions is constant.

Existence

- Nummelin (1984), Kontoyiannis and Meyn (2003, 2005)
- Recurrence of kernels
- Liapunov functions