The Uncertainty Premium in an Ambiguous Economy

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Ambiguity - Example

Lottery A
- A basket contains 50 black balls and 50 red balls
- Red ball = $2
- Black ball = $0

Price the bet on pulling a red ball.

Lottery B
- A basket contains 100 balls: black balls and red balls
- Red ball = $2
- Black ball = $0

What would be the price of a bet on pulling a red ball?
Ambiguity vs. Risk

- **Risk** is a situation in which the relative odds of events are known.
- **Ambiguity** is a condition in which the probabilities of events are either not uniquely assigned or are unknown.
Main Target

- Investigate asset pricing under ambiguity:
  - Can ambiguity be referred as extra risk?
  - Can the equity premium puzzle be explained by ambiguity?
Outline

- Literature review
- Overview Klibanoff, Marinacci and Mukerji (KMM, 2005) model
- Ambiguous economy construction (KMM based)
- Analysis of KMM economy
- Intuition
- Conclusions
Classical Utility Theories

The fundamental assumption of both the von Neumann-Morgenstern (1944) and Savage (1954) theories is that agents with an expected utility representation know, or act as if they know, the probabilities of all states.
Early Research

- Knight (1921) - Distinction between risk and ambiguity.
- Ellsberg (1961) - Individuals prefer to choose when the probabilities are known, and that they are willing to pay in order to avoid choosing in an ambiguous context (Ellsberg paradox).
KMM Model

- Klibanoff, Marinacci and Mukerji (2005, KMM)
- Generalization of max-min expected utility model with multiple priors.
- The agent has a subjective probability distributions over a set of possible priors.
KMM Model

The agent preferences take the double expectation form:

- \( u \) – attitude toward risk
- \( \phi \) – attitude toward ambiguity

\[
V(f) = \int_{P} \phi \left( \int_{S} u(f) \, d\pi \right) \, d\psi = E_{\psi} \phi \left( E_{\Pi} u(f) \right)
\]
The agent orders her preferences over random consumption paths by

\[ V(c_t) = u(c_t) + \delta \phi^{-1} \left[ \int_P \phi \left( \int_S V(\mathcal{C}_{t+1}) \, d\pi \right) \, d\psi \right] \]

\[ = u(c_t) + \delta \phi^{-1} \left[ E_{\Psi, \tau} \phi \left( E_{\Pi, \tau} V(\mathcal{C}_{t+1}) \right) \right] \]
Binomial Economy

- Two states of world
  - $d$ (down) the “bad” state.
  - $u$ (up) the “good” state.
- Two probability distributions:
  - $P = \{(\pi_A, 1-\pi_A), (\pi_B, 1-\pi_B)\}$
- Probability distributions $A$ is better then $B$
  - $\pi_A \geq \pi_B$
Binomial Economy

- The probability over the set of prior is
  \[
  \Psi = (\psi, 1-\psi)
  \]

- Agent I is more pessimist than agent II if she assigns higher probability to the “bad” distribution:
  \[
  \psi_I \leq \psi_{II}
  \]
Binomial Economy

- Ambiguity degree - the difference between the high (good) probability of the good state and the low (bad) probability of that state.

\[ \Delta = \pi_A - \pi_B \geq 0 \]
\begin{align*}
\psi & \quad \pi_A \quad u \\
1 - \psi & \quad 1 - \pi_A \quad d
\end{align*}
\begin{align*}
\psi & \quad \pi_B \quad u \\
1 - \psi & \quad 1 - \pi_B \quad d
\end{align*}
\begin{align*}
\psi \pi_A + (1 - \psi) \pi_B \\
\psi (1 - \pi_A) + (1 - \psi)(1 - \pi_B)
\end{align*}
Risk Premium

Pratt (1964): 

\[ \text{Risk premium} = \rho = -\frac{\sigma^2}{2} \frac{u''(\bar{c})}{u'(\bar{c})} = \text{ARA} \frac{\sigma^2}{2} \]

Derived from certainty equivalence:

\[ u(\bar{c} - \rho) = E \left[ u(\delta) \right] \]
Uncertainty Premium - Preliminaries

- Expected consumption
  \[ \bar{c} = E_\Psi \left[ E_\Pi \left( \phi \right) \right] \]

- Variance of consumption
  \[ \sigma^2 = E_\Psi \left[ E_\Pi \left( \phi - \bar{c} \right)^2 \right] \]

- Variance of expected consumption
  \[ \sigma^2_\mu = E_\Psi \left[ E_\Pi \left( \phi - \bar{c} \right) \right]^2 \]
Uncertainty Premium, $\kappa$

- The Uncertainty Premium, $\kappa$, is derived from certainty equivalence:

$$E_\Psi \left[ \phi \left( E_\Pi u \left( \bar{c} - \kappa \right) \right) \right] = E_\Psi \left[ \phi \left( E_\Pi u \left( \phi \right) \right) \right]$$
Uncertainty Premium, $\kappa$ (Theorem 1)

The uncertainty premium $\kappa$ is the sum of the ambiguity and risk premium:

$$\kappa = - \frac{\sigma^2}{\mu} \phi''(u(\overline{c})) u'(\overline{c}) \frac{2}{4} \frac{\phi'(u(\overline{c}))}{4} \frac{4}{4} \frac{2}{4} u'(\overline{c})$$

**Ambiguity Premium**

**Risk Premium**
Constant Relative Risk/Ambiguity Aversion (Corollary)

Preferences

\[ u(c, \gamma) = \begin{cases} 
    c^{1-\gamma}, & \gamma < 1 \\
    \ln(c), & \gamma = 1 
\end{cases} \]

Uncertainty premium:

\[ \kappa = (1 - \gamma) \left[ \eta \frac{\sigma^2}{2c^2} \right] + \gamma \frac{\sigma^2}{2c} \]
Uncertainty Premium (Theorem 4)

- In an ambiguous economy, CRRA, CRAA, and a risk aversion coefficient $0 < \gamma < 1$,
  - the uncertainty premium $\uparrow$
  - when the ambiguity aversion $\uparrow$. 
Uncertainty Premium for Increasing Ambiguity Aversion

Uncertainty Premium for Increasing Eta. In this graph, Gamma = 0.5
Uncertainty Premium (Theorem 5 and 6)

- In an ambiguous economy, CRRA and CRAA:
  - For ambiguity aversion
    - the uncertainty premium $\uparrow$
    - when the risk aversion $\uparrow$
  - For ambiguity aversion
    - the uncertainty premium $\downarrow$
    - when the risk aversion $\uparrow$

\[ 0 \leq \eta \leq \frac{\sigma^2}{\sigma^2_{\mu}} \]
\[ \eta > \frac{\sigma^2}{\sigma^2_{\mu}} \]
Ambiguity Premium (Theorem 3)

- In an ambiguous economy, CRRA, CRAA
  - the ambiguity premium ↓
  - when the risk aversion ↑
Uncertainty Premium for Increasing Risk Aversion

Uncertainty Premium for Increasing Gamma. In this graph, Eta = 2.0
Uncertainty Premium for Increasing Risk Aversion

Uncertainty Premium for Increasing Gamma. In this graph, Eta = 4.0
Intuition

- Individuals prefer to choose when the probabilities are known and they are willing to pay in order to avoid choosing in ambiguous context (Ellsberg, 1961)
Intuition

- When circumstances are worse (higher degree of ambiguity, higher pessimism, higher risk aversion and higher ambiguity aversion):
  - Uncertainty premium increases
  - State prices decreases
  - Risk-free rate decreases
The ambiguity premium is the premium that the agent is willing to pay in order to avoid the dispersion of expected utilities.

In an ambiguous economy, CRRA and CRAA:

\[
\text{Ambiguity Premium} = (1 - \gamma) \left[ \eta \frac{\sigma^2}{\mu} \right]
\]
Ambiguity Premium (Theorem 3)

- In an ambiguous economy, CRRA, CRAA
  - the ambiguity premium ↓
  - when the risk aversion ↑
- Increasing risk aversion results in lower dispersion of expected utilities → lower ambiguity premium.
Dispersion of Expected Utilities

Expected Utility as a Function of Probability Measure

Risk aversion, $\gamma$

Expected utility

{0.3,0.7}

{0.7,0.3}
Equity Premium Puzzle

- Does ambiguity act as extra risk?, might it explain the equity premium puzzle?
  - Epstein and Wang (1994)
  - Hansen, Sargent and Tallerini (1999)
  - Chen and Epstein (2002)
  - Klibanoff, Marinacci and Mukerji (2005)
  - Gollier (2005)
Conclusion

- The uncertainty premium is higher than the risk premium
- The ambiguity premium is positive
- Ambiguity might give additional explanation to the “equity premium puzzle”
- Ambiguity not necessarily act as extra risk
End
Risk-Free Rate

- Target - pricing assets using state prices
- The risk-free asset represents the “average” state prices.
State Prices – Classical Economy

\[ q_u = \delta \pi \frac{\partial u'(uC_0)}{\partial u'(C_0)} \]

\[ q_d = \delta (1 - \pi) \frac{\partial u'(dC_0)}{\partial u'(C_0)} \]
State Prices – Ambiguous Economy

\[ q_u = \delta \frac{u'(uC_0)}{u'(C_0)} \{ \psi \pi_A \phi'(E_A) + (1-\psi) \pi_B \phi'(E_B) \} (\phi^{-1})'(E) \]

\[ q_d = \delta \frac{u'(dC_0)}{u'(C_0)} \{ \psi (1-\pi_A) \phi'(E_A) + (1-\psi) (1-\pi_B) \phi'(E_B) \} (\phi^{-1})'(E) \]

\[ E = E_\psi \phi(E_\Pi u(g)) \]
Risk-Free Rate Under KMM Model

- For a given ambiguity level, increasing pessimism results in a lower risk-free rate.
- A greater degree of ambiguity results in a lower risk-free rate.
Risk-Free Rate Under KMM Model

- An increase in ambiguity aversion leads to a decrease in the risk-free rate.
- An increase in risk aversion leads to an increase in the risk-free rate.
- The risk-free rate in an economy with ambiguity aversion but with no ambiguity is collapse to classical risk-free rate.
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<thead>
<tr>
<th></th>
<th>Uncertainty premium</th>
<th>Risk-free rate</th>
<th>State prices</th>
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</thead>
<tbody>
<tr>
<td>Ambiguity aversion</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Risk aversion</td>
<td>+/-</td>
<td>+</td>
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<tr>
<td>Ambiguity degree</td>
<td>?</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Pessimism</td>
<td>?</td>
<td>-</td>
<td>+</td>
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