Math Finan Econ (2008) 1:213–249
DOI 10.1007/s11579-008-0009-3

Asymptotic arbitrage and large deviations

H. Föllmer · W. Schachermayer

Received: 30 May 2007 / Accepted: 21 April 2008 / Published online: 21 May 2008

Abstract Typical models of mathematical finance admit equivalent martingale measures up to any finite time horizon but not globally, and this means that arbitrage opportunities arise in the long run. In this paper, we derive explicit estimates for asymptotic arbitrage, and we show how they are related to large deviation estimates for the market price of risk. As a case study we consider a geometric Ornstein–Uhlenbeck process. In this setting we also compute the optimal trading strategies and the resulting optimal growth rates of expected utility for all HARA utilities.

Keywords Asymptotic arbitrage · Utility maximization · Large deviations · Cost averaging

JEL Classification G11 · G12 · C61

1 Introduction

In this paper, we investigate the issue of optimizing terminal wealth by investing in a financial market, when the time horizon T tends to infinity. We focus on the following issues:

(i) A better understanding of the features of the financial market $(S_t)_{t\geq 0}$ which insure exponential growth of the terminal wealth X_T of an investor in this market, for $T \to \infty$.

W. Schachermayer gratefully acknowledges Financial support from the Austrian Science Fund (FWF) under the grant P19456, from Vienna Science and Technology Fund (WWTF) under Grant MA13 and by the Christian Doppler Research Association (CDG).

H. Föllmer
Institut für Mathematik, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany

W. Schachermayer (✉)
Vienna University of Technology, Wiedner Hauptstrasse 8-10/105, 1040 Wien, Austria
e-mail: wschach@fam.tuwien.ac.at

(ii) Relating this question to the notion of asymptotic arbitrage, to the theory of large deviations, and to utility maximization with respect to the power utilities $U(x) = \frac{x^\alpha}{\alpha}$, for $\alpha \in \,]-\infty, 0[$.
(iii) Analyzing carefully the case study of a geometric Ornstein–Uhlenbeck process and comparing it to the well-known situation of the Black–Scholes model by explicitly calculating the optimal strategies for HARA-utility optimizers.

We consider an $\mathbb{R}^d$-valued semi-martingale $S = (S_t)_{t \geq 0}$ modeling the price process of d risky assets with infinite horizon. The bond is assumed to be normalized by $B_t \equiv 1$, i.e., we consider S in discounted terms. For a fixed finite horizon T, let

$$K_T = \{(H \cdot S)_T \mid H \in \mathcal{H}\},$$

denote the set of attainable contingent claims, where $\mathcal{H}$ denotes the class of predictable, S-integrable, admissible processes (for a definition see e.g. [7, Definition 8.1.1, p. 130]).

We recall the notion of *strong asymptotic arbitrage* as introduced by Kabanov and Kramkov [19] which we specialize to the present situation of a varying time horizon T.

Definition 1.1 The process $S = (S_t)_{t \geq 0}$ allows for *strong asymptotic arbitrage* if, for $\varepsilon > 0$, there is $T < \infty$ and $X_T \in K_T$ satisfying

(i) $X_T \geq -\varepsilon$, a.s., and
(ii) $\mathbf{P}[X_T \geq \varepsilon^{-1}] \geq 1 - \varepsilon$.

The economic interpretation is straightforward: condition (i) means that the maximal loss of the trading strategy, yielding the wealth X_T at time T, is bounded by ε; condition (ii) means that with probability $1 - \varepsilon$ the terminal wealth X_T equals at least ε^{-1}.

In [19] a dual characterization of this concept was given in terms of the Hellinger distance of the equivalent martingale measures for the process $(S_t)_{0 \leq t \leq T}$ to the original measure $\mathbf{P}$ (compare also [22]). In Proposition 2.1 we take up this theme again and relate these Hellinger distances to utility maximization for power utility $U(x) = \frac{x^\alpha}{\alpha}$, where $\alpha \in \,]-\infty, 0[$.

For the sake of clarity of exposition we shall focus on the setting of diffusions driven by Brownian motion; we note however, that many of the results below could be extended to more general situations.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ be a filtered probability space such that $(\mathcal{F}_t)_{t \geq 0}$ is the (right continuous, saturated) filtration generated by an $\mathbb{R}^N$-valued standard Brownian motion $(W_t)_{t \geq 0}$.

Assumption 1.2 The process $S = (S_t)_{t \geq 0}$ will be assumed to be an $\mathbb{R}^d$-valued diffusion based on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ such that there is a (deterministic, time-independent) volatility function

$$\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times N} \tag{1}$$

as well as a market price of risk function

$$\varphi : \mathbb{R}^d \to \mathbb{R}^N \tag{2}$$

such that

$$dS_t = \sigma(S_t)(dW_t + \varphi(S_t)dt), \tag{3}$$

where we may and do suppose that $\varphi(S_t)$ takes its values in $\ker(\sigma(S_t))^\perp$.

We also assume that

$$Z_t^{\min} = \exp\left[-\int_0^t (\varphi_u, dW_u) - \frac{1}{2}\int_0^t \|\varphi_u\|^2 du \right] \tag{4}$$

is a strictly positive martingale, where we put $\varphi_u = \varphi(S_u)$ and where $(\,.\,,\,.\,)$ and $\|\,.\,\|$ denote the inner product and the corresponding Euclidean norm on $\mathbb{R}^N$.

We then deduce from Girsanov's formula that, for each $T \geq 0$, the measure $\mathbf{Q}_T^{\min}$ on $\mathcal{F}_T$ defined by

$$\frac{d\mathbf{Q}_T^{\min}}{d\mathbf{P}} = Z_T^{\min}$$

is a probability measure equivalent to the restriction of $\mathbf{P}$ to $\mathcal{F}_T$ such that $(S_t)_{0\leq t\leq T}$ is a local martingale under $\mathbf{Q}_T^{\min}$. Following [15] we call $\mathbf{Q}_T^{\min}$ the *minimal* martingale measure for $(S_t)_{0\leq t\leq T}$.

It seems worthwile to comment on the above concepts. Suppose for simplicity that φ and σ are constant. If the $d \times N$ matrix σ is injective, then the law of the process S in (3) uniquely determines the vector $\varphi \in \mathbb{R}^N$; this corresponds to the case of a complete market. On the other hand, in the incomplete case, i.e. for a non-injective matrix σ, the vector φ is only determined by the process S in (3) up to adding elements in the kernel of σ. There is one canonical choice of φ, namely the one orthogonal to $\ker(\sigma)$. The fact that this choice of φ has minimal norm in $\mathbb{R}^N$ motivates the name "minimal" above.

A central question of our present investigation will be to understand which features of the above model imply that there is exponential growth of a well chosen attainable portfolio as the time horizon T goes to ∞. It is obvious that one has to impose some assumption on the market price of risk. Indeed, if φ vanishes then the process $(S_t)_{t\geq 0}$ is a local martingale, and one cannot systematically win by betting on a local martingale in an admissible way.

Definition 1.3 Under the above assumptions we say that the diffusion process $S = (S_t)_{t\geq 0}$ has *an average squared market price of risk above the threshold* $c > 0$ if the process $(\|\varphi_t\|)_{t\geq 0}$ satisfies the following estimate:

$$\lim_{T\to\infty} \mathbf{P}\left[\frac{1}{T}\int_0^T \|\varphi_t\|^2 dt < c \right] = 0. \tag{5}$$

If there is $c > 0$ such that (5) holds true we say that S has *a non-trivial market price of risk*.

We say that *the market price of risk satisfies a large deviations estimate* if there are constants $c_1, c_2 > 0$ such that

$$\limsup_{T\to\infty} \frac{1}{T} \log\left(\mathbf{P}\left[\frac{1}{T}\int_0^T \|\varphi_t\|^2 dt \leq c_1 \right] \right) < -c_2. \tag{6}$$

The economic interpretation of (5) is that the market price of risk should on average be bounded away from zero in the long run. This assumption is satisfied whenever the diffusion is ergodic with invariant measure μ and if the market price of risk function φ is not μ-a.s. equal to 0, since ergodicity implies

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \|\varphi_t\|^2 dt = \int_{\mathbb{R}^d} \|\varphi(x)\|^2 \mu(dx) \quad \mathbf{P}\text{-a.s.}$$

As explained in Sect. 3, the large deviations estimate (6) for the market price of risk follows by a contraction principle whenever the diffusion process $(S_t)_{t\geq 0}$ is ergodic and satisfies a principle of large deviations. We refer to the large deviations literature for suitable conditions; see, e.g., [8]. For the Black–Scholes model the market price of risk is constant, and so the large deviations estimate (6) is trivially satisfied if the market price of risk is assumed to be different from zero. Less trivial is the example of the geometric Ornstein–Uhlenbeck process which will be analyzed in Sects. 4 and 5, using large deviations results from [14,28]); see also [13]. In this example the large deviations estimate (6) holds true, and the optimal pairs (c_1, c_2) can be calculated explicitly.

Assumption (5) is of course weaker than (6), but it is sufficient to deduce the following estimates which will be proved in Sect. 3.

Theorem 1.4 *Let $S = (S_t)_{t\geq 0}$ be a process satisfying Assumption 1.2 and having an average squared market price of risk above the threshold $c > 0$; cf. (5).*

For $\varepsilon > 0$, $\gamma_1 + \gamma_2 < c/2$, and for T large enough, there exists $X_T \in K_T$ such that

(i) $X_T \geq -e^{-\gamma_1 T}$,
(ii) $\mathbf{P}\left[X_T \geq e^{\gamma_2 T}\right] \geq 1 - \varepsilon$.

The message of the theorem is that the assumption of a non-trivial market price of risk (5) implies asymptotic arbitrage; in fact we obtain exponential estimates for the maximal loss in (i) as well as for the "typical" growth in (ii).

If Assumption (5) is replaced by the stronger large deviation estimate (6), one should even expect an exponential decay in time for the probability of falling short of the exponential lower bound in assertion (ii) above. This will be discussed in Sect. 3. In Sect. 4 we illustrate the situation for the geometric Ornstein–Uhlenbeck process defined by

$$S_t = \exp(Y_t), \tag{7}$$

where $(Y_t)_{t\geq 0}$ denotes an Ornstein–Uhlenbeck process defined by

$$dY_t = -\rho Y_t dt + \sigma dW_t, \quad Y_0 = y_0, \tag{8}$$

for constants $\sigma > 0$, $\rho > 0$, and $y_0 \in \mathbb{R}$. Here the market price of risk $\varphi_t = \varphi(S_t)$ is given by

$$\varphi_t = -\frac{\rho}{\sigma} Y_t + \frac{\sigma}{2}$$

(see (14)), and we obtain the subsequent explicit results:

Theorem 1.5 *Let $(S_t)_{t\geq 0}$ be a geometric Ornstein–Uhlenbeck process as in (7). For any $\gamma, \gamma_1, \gamma_2 \in \mathbb{R}$ such that $\gamma_1 + \gamma_2 < \gamma \in \left]0, \frac{\sigma^2}{8} + \frac{\rho}{4}\right[$, there exist attainable contingent claims $X_T \in K_T$ such that*

(i) $X_T \geq -e^{-\gamma_2 T}$,
(ii) $\lim_{T\to\infty} \frac{1}{T} \log \mathbf{P}\left[X_T < e^{\gamma_1 T}\right] = \frac{-\left(\frac{\sigma^2}{8} - \gamma + \frac{\rho}{4}\right)^2}{\frac{\sigma^2}{8} - \gamma + \frac{\rho}{2}}$.

In fact we are going to prove a stronger version, with sharper bounds in (i) and a corresponding optimality result for the rate of convergence in (ii); cf. Theorem 4.2 and Remark 4.3.

Remark 4.4 describes the extension to an Ornstein–Uhlenbeck process with drift. A corresponding result for the Black–Scholes model is given in Theorem 4.5.

How are these results, which ensure "asymptotic arbitrage" in a rather strong sense, related to the theme of utility maximization? For the case of logarithmic utility, Theorem 1.4 easily implies a lower bound for the attainable expected utility per unit time; see Proposition 3.2. In fact, there is a deeper connection of the results on asymptotic arbitrage and utility maximization. In Proposition 2.3 we give rather sharp estimates between asymptotic arbitrage and utility maximization with respect to the power utilities $U(x) = \frac{x^\alpha}{\alpha}$, where α ranges in $]-\infty, 0[$. This result is interesting in its own right and also allows for a better understanding of the duality theory of asymptotic arbitrage [19,22].

In Sect. 5 we continue our case study of the geometric Ornstein–Uhlenbeck process and calculate explicitly the optimal trading strategies and the resulting expected utility $u_T(x)$ for all $T > 0$ and all HARA-utilities. As a corollary we obtain the optimal growth rates for $u_T(x)$. In the case of power utility, this may be viewed as a probabilistic complement to the dynamic programming approach in Fleming and Sheu [13], as explained in Remark 5.8. While for logarithmic utility there are no surprises and our findings are similar to the wellknown results of R. Merton in the case of the Black–Scholes model [26,27], we find some counter-intuitive results in the case of power utility $U(x) = \frac{x^\alpha}{\alpha}$, for $\alpha \in]-\infty, 1[\setminus \{0\}$, and of exponential utility $U(x) = -\exp(-\lambda x)$.

2 Asymptotic arbitrage and power utility

We start with an easy but surprisingly sharp dual characterization of the notion of asymptotic arbitrage. In this proposition we take the horizon T as fixed.

Proposition 2.1 *Let $S = (S_t)_{0 \le t \le T}$ be an $\mathbb{R}^d$-valued locally bounded semi-martingale such that*

$$\mathcal{M}_T^e(S) = \{\mathbf{Q} \sim \mathbf{P} \mid S \textit{ is a local } \mathbf{Q}\textit{-martingale}\} \neq \emptyset.$$

For $1 > \varepsilon_1, \varepsilon_2 > 0$ the statements (a) and (b) are equivalent:

(a) *There is $X_T \in K_T$ such that*
  (i) $X_T \ge -\varepsilon_2$,
  (ii) $\mathbf{P}[X_T \ge 1 - \varepsilon_2] \ge 1 - \varepsilon_1$.

(b) *There is $A_T \in \mathcal{F}_T$ with $\mathbf{P}[A_T] \le \varepsilon_1$ such that, for each $\mathbf{Q} \in \mathcal{M}_T^e(S)$ we have $\mathbf{Q}[A_T] \ge 1 - \varepsilon_2$.*

In this case we say that S admits an $(\varepsilon_1, \varepsilon_2)$-arbitrage (up to time T).

Proof (a) $\Rightarrow$ (b): Let $A_T = \{X_T < 1 - \varepsilon_2\}$ so that $\mathbf{P}[A_T] \le \varepsilon_1$, and take $\mathbf{Q} \in \mathcal{M}_T^e(S)$. Since $X_T \in K_T$ implies $\mathbf{E_Q}[X_T] \le 0$ (see, e.g., [5, Theorem 5.7]), we may estimate

$$(1 - \varepsilon_2)\mathbf{Q}[\Omega \backslash A_T] - \varepsilon_2 \mathbf{Q}[A_T] \le \mathbf{E_Q}[X_T] \le 0,$$

hence $\mathbf{Q}[A_T] \ge 1 - \varepsilon_2$.

(b) $\Rightarrow$ (a): If A_T satisfies the assumptions of (b) then

$$X_T = -\varepsilon_2 \mathbf{1}_{A_T} + (1 - \varepsilon_2)\mathbf{1}_{\Omega \backslash A_T}$$

has properties (i) and (ii) and satisfies $\mathbf{E_Q}[X_T] \le 0$ for each $\mathbf{Q} \in \mathcal{M}_T^e(S)$. Applying again the superhedging theorem [5, Theorem 5.7] we see that X_T is dominated by an element $\widetilde{X}_T \in K_T$, and $\widetilde{X}_T$ clearly inherits properties (i) and (ii). □

We remark that the assumption of local boundedness is not really relevant in the present context. It could be dropped by replacing the concept of local martingales by the concept of sigma-martingales (see [6]).

In the next result we relate strong asymptotic arbitrage with dynamic portfolio optimisation for a certain class of utility functions which includes the power utilities $U(x) = \frac{x^\alpha}{\alpha}$ for $-\infty < \alpha < 0$ as typical examples.

Proposition 2.2 *Let $S = (S_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued locally bounded semi-martingale such that, for each $T > 0$, the set $\mathcal{M}_T^e(S)$ is not empty. Let $U : \mathbb{R}_+ \to \mathbb{R}$ be a strictly increasing concave function such that*

$$\lim_{x\searrow 0} U(x) = -\infty, \quad \lim_{x\to\infty} U(x) = 0.$$

The following assertions are equivalent:

(a) *$(S_t)_{t\geq 0}$ allows for strong asymptotic arbitrage.*

(b) *Defining the value function*

$$u_T(x) = \sup_{X_T\in K_T} \mathbf{E}[U(x + X_T)],$$

we have

$$\lim_{T\to\infty} u_T(x) = 0,$$

for some $x > 0$ (or, equivalently, for all $x > 0$).

Proof (a) $\Rightarrow$ (b): Given $x > 0$, let $0 < \varepsilon < \frac{x}{2}$. If $X_T \in K_T$ satisfies (i) and (ii) of Definition 1.1 we get

$$\begin{aligned} u_T(x) &\geq \mathbf{E}[U(x + X_T)] \\ &\geq U\left(\frac{x}{2}\right) P[X_T < \varepsilon^{-1}] + U(\varepsilon^{-1}) P[X_T \geq \varepsilon^{-1}] \\ &\geq \varepsilon U\left(\frac{x}{2}\right) + U(\varepsilon^{-1}). \end{aligned}$$

Since the latter expression tends to zero as $\varepsilon \to 0$, we obtain $\lim_{T\to\infty} u_T(x) = 0$.

(b) $\Rightarrow$ (a): Suppose that there is $x > 0$ such that $\lim_{T\to\infty} u_T(x) = 0$. Thus we may find $X_T \in K_T$ such that $X_T \geq -x$ and

$$\lim_{T\to\infty} \mathbf{E}[U(x + X_T)] = 0.$$

Take $\varepsilon > 0$. Applying Tschebyscheff's inequality in the form

$$\begin{aligned} \mathbf{P}[X_T < x\varepsilon^{-2}] &= \mathbf{P}[|U(x + X_T)| > |U(x + x\varepsilon^{-2})|] \\ &\leq |U(x + x\varepsilon^{-2})]|^{-1} |E[U(x + X_T)]|, \end{aligned}$$

we obtain $T_\varepsilon > 0$ such that

$$\mathbf{P}[X_T \geq x\varepsilon^{-2}] \geq 1 - \varepsilon$$

for $T \geq T_\varepsilon$. Since $X_T \geq -x$ we conclude that $\varepsilon X_T / x$ satisfies the requirements of Definition 1.1. □

The preceding result is of a purely qualitative nature. Since we also want to obtain quantitative results on the speed of convergence, we now specialize to the case of power utility

$$U^{(\alpha)}(x) = \frac{x^\alpha}{\alpha}, \quad -\infty < \alpha < 0.$$

In this case the conjugate function

$$V(y) = \sup_{x>0}[U^{(\alpha)}(x) - xy], \quad y > 0,$$

is given by

$$V(y) = V^{(\beta)}(y) = -\frac{y^\beta}{\beta}$$

where $\beta = \frac{\alpha}{\alpha-1} \in]0, 1[$.

As usual in utility optimization we write for the primal and dual value functions [24]

$$u_T^{(\alpha)}(x) = \sup_{X_T \in K_T} \mathbf{E}[U^{(\alpha)}(x + X_T)], \quad x > 0,$$
$$v_T^{(\beta)}(y) = \inf_{\mathbf{Q} \in \mathcal{M}_T^e(S)} \mathbf{E}\left[V^{(\beta)}\left(y\frac{d\mathbf{Q}}{d\mathbf{P}}\right)\right], \quad y > 0.$$

We obtain from the scaling property of the power function that we have $u_T^{(\alpha)}(x) = c_T U^{(\alpha)}(x)$, for some $0 \leq c_T \leq 1$, as well as $v_T^{(\beta)}(y) = c_T^* V^{(\beta)}(y)$, where $c_T^* = c_T^{\frac{1}{|\alpha|+1}}$ (compare [24]).

The following proposition is similar to the results in [19] in terms of the concept of Hellinger distance, but there it is not connected with the idea of utility maximization. Again the horizon T will be fixed.

Proposition 2.3 *Let $-\infty < \alpha < 0$, $\beta = \frac{\alpha}{\alpha-1}$, and an $\mathbb{R}^d$-valued semi-martingale $S = (S_t)_{0 \leq t \leq T}$ as in Proposition 2.1 be given. For $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$ consider the following statements.*

- (i) *S admits an $(\varepsilon_1, \varepsilon_2)$-arbitrage.*
- (i') *There is $A \in \mathcal{F}_T$, $\mathbf{P}[A] \leq \varepsilon_1$, such that, for each $\mathbf{Q} \in \mathcal{M}_T^e(S)$ we have $\mathbf{Q}[A] \geq 1 - \varepsilon_2$.*
- (ii) *For each $\mathbf{Q} \in \mathcal{M}_T^e(S)$ there is $A \in \mathcal{F}_T$, such that $\mathbf{P}[A] \leq \varepsilon_1$ and $\mathbf{Q}[A] \geq 1 - \varepsilon_2$.*
- (iii) *$u_T^{(\alpha)}(x) \geq \varepsilon U^{(\alpha)}(x)$, for some (or, equivalently, for all) $x > 0$.*
- (iii') *$v_T^{(\beta)}(y) \geq \varepsilon^{\frac{1}{|\alpha|+1}} V^{(\beta)}(y)$, for some (or, equivalently, for all) $y > 0$.*

Then the following assertions hold true:

- (a) *(i) $\Leftrightarrow$ (i') and (iii) $\Leftrightarrow$ (iii').*
- (b) *(i') $\Rightarrow$ (ii). The reverse implication (ii) $\Rightarrow$ (i') holds true if $\mathcal{M}_T^e(S) = \{\mathbf{Q}\}$ is a singleton (complete financial market).*
- (c) *(ii) $\Rightarrow$ (iii) if $\varepsilon \geq 2^{1+|\alpha|} \max(\varepsilon_1, \varepsilon_2^{|\alpha|})$.*
- (d) *(iii) $\Rightarrow$ (i) if $\varepsilon \leq \varepsilon_1 \varepsilon_2^{|\alpha|}$.*

Proof (a) follows from Proposition 2.1 and the discussion preceding Proposition 2.3, while (b) is obvious.

(c): Fix $\mathbf{Q} \in \mathcal{M}_T^e(S)$ and the corresponding set $A \in \mathcal{F}_T$ satisfying $\mathbf{P}[A] \leq \varepsilon_1$ and $\mathbf{Q}[A] \geq 1 - \varepsilon_2$ as in (ii). By possibly passing to smaller values of $\varepsilon_1, \varepsilon_2$, we may suppose that $\mathbf{P}[A] = \varepsilon_1$ and $\mathbf{Q}[A] = 1 - \varepsilon_2$. In view of (a) it is enough to verify (iii'). By Jensen we obtain

$$\begin{aligned}\mathbf{E}\left[V^{(\beta)}\left(\tfrac{d\mathbf{Q}}{d\mathbf{P}}\right)\right] &= -\frac{1}{\beta}\mathbf{E}\left[\left(\tfrac{d\mathbf{Q}}{d\mathbf{P}}\right)^{\beta}\right] \\ &\geq -\frac{1}{\beta}\mathbf{E}\left[\mathbf{1}_A\left(\frac{\mathbf{Q}[A]}{\mathbf{P}[A]}\right)^{\beta} + \mathbf{1}_{\Omega\setminus A}\left(\frac{\mathbf{Q}[\Omega\setminus A]}{\mathbf{P}[\Omega\setminus A]}\right)^{\beta}\right] \\ &= -\frac{1}{\beta}\left[\varepsilon_1^{1-\beta}(1-\varepsilon_2)^{\beta} + (1-\varepsilon_1)^{1-\beta}\varepsilon_2^{\beta}\right] \\ &\geq -\frac{2}{\beta}\max\left(\varepsilon_1^{1-\beta}, \varepsilon_2^{\beta}\right).\end{aligned}$$

As this inequality holds true for all $\mathbf{Q} \in \mathcal{M}_T^e(S)$ we obtain

$$v_T^{(\beta)}(1) \geq 2\max\left(\varepsilon_1^{1-\beta}, \varepsilon_2^{\beta}\right) V^{(\beta)}(1).$$

Using $\frac{1}{|\alpha|+1} = \frac{-1}{\alpha-1} = 1 - \beta$ we conclude from the assumption

$$\begin{aligned}\varepsilon^{\frac{1}{|\alpha|+1}} &\geq 2\max\left(\varepsilon_1^{\frac{1}{|\alpha|+1}}, \varepsilon_2^{\frac{|\alpha|}{|\alpha|+1}}\right) \\ &= 2\max\left(\varepsilon_1^{1-\beta}, \varepsilon_2^{\beta}\right)\end{aligned}$$

that

$$v_T^{(\beta)}(1) \geq \varepsilon^{\frac{1}{|\alpha|+1}} V^{(\beta)}(1),$$

which yields (iii').

(d): Let $\varepsilon, \varepsilon_1, \varepsilon_2$ satisfy $\varepsilon < \varepsilon_1\varepsilon_2^{|\alpha|}$. By assumption (iii) there is $X_T \in K_T$ such that

$$\mathbf{E}[U^{(\alpha)}(\varepsilon_2 + X_T)] \geq \varepsilon U^{(\alpha)}(\varepsilon_2) = \varepsilon\frac{\varepsilon_2^{\alpha}}{\alpha}.$$

Clearly $X_T \geq -\varepsilon_2$ a.s. In order to verify (i) it remains to show that $\mathbf{P}[X_T \geq 1-\varepsilon_2] \geq 1-\varepsilon_1$. Indeed, using Tschebyscheff we obtain

$$\mathbf{E}[U^{(\alpha)}(\varepsilon_2 + X_T)] \leq \mathbf{P}[X_T < 1 - \varepsilon_2]\, U^{(\alpha)}(1),$$

hence

$$-\varepsilon\frac{\varepsilon_2^{\alpha}}{\alpha} \geq \mathbf{P}[X_T < 1 - \varepsilon_2]\left(-\frac{1}{\alpha}\right),$$

and therefore

$$\varepsilon_1 > \mathbf{P}[X_T < 1 - \varepsilon_2].$$

□

Remark 2.4 Statements (i') and (ii) above only differ in the order of the quantifiers, so that we have the trivial implication (i') ⇒ (ii), as well as (ii) ⇒ (i') in the case when $\mathcal{M}_T^e(S)$ is a singleton.

The more interesting part of Proposition 2.3 is that we may also conclude that (ii) ⇒ (i') in the incomplete case, provided we replace the constants $\varepsilon_1, \varepsilon_2$ in (i') by bigger constants

$\widetilde{\varepsilon}_1, \widetilde{\varepsilon}_2$. For example we may choose $\widetilde{\varepsilon}_1 = 2^{1+|\alpha|}\varepsilon_1^{\frac{1}{2}}, \widetilde{\varepsilon}_2 = \varepsilon_2^{\frac{1}{2}}$, where $\alpha \in]-\infty, 0[$ satisfies $\varepsilon_1 = \varepsilon_2^{|\alpha|}$. Indeed, for given $\varepsilon_1, \varepsilon_2 > 0$ and $\alpha \in]-\infty, 0[$ such that $\varepsilon_1 = \varepsilon_2^{|\alpha|}$, we have (ii) $\Rightarrow$ (iii'), if we let $\varepsilon = 2^{1+|\alpha|}\varepsilon_1$. We then have $\varepsilon \leq \widetilde{\varepsilon}_1\widetilde{\varepsilon}_2^{|\alpha|} = 2^{1+|\alpha|}\varepsilon_1^{\frac{1}{2}}\varepsilon_1^{\frac{1}{2}}$ so that we obtain (iii) $\Rightarrow$ (i) $\Leftrightarrow$ (i'), if we use in (i) and (i') the constants $\widetilde{\varepsilon}_1, \widetilde{\varepsilon}_2$ instead of $\varepsilon_1, \varepsilon_2$.

Summing up, we can reverse the quantifiers in statement (ii) provided we content ourselves with somewhat worse constants. This phenomenon is related to a quantitative version of the Halmos–Savage theorem, as was stressed in [22].

3 Estimates for asymptotic arbitrage

In this section we show under suitable regularity conditions that price processes with a non-trivial market price of risk (5) allow for asymptotic arbitrage; more precisely, we prove the estimates of Theorem 1.4. We also show how these estimates can be refined if the market price of risk satisfies a large deviations estimate.

Let us fix a diffusion process S satisfying Assumption 1.2, and recall the minimal martingale measure $\mathbf{Q}_T^{\min}$ for $(S_t)_{0\leq t\leq T}$ defined via (4) .

It follows from [15] that, for an arbitrary $\mathbf{Q}_T \in \mathcal{M}_T^e(S)$ there is a predictable process $(\psi_t)_{0\leq t\leq T}$ such that the density $Z_T = \frac{d\mathbf{Q}_T}{d\mathbf{P}}$ is given by

$$Z_T = \exp\left[\int_0^T \left(-\psi_t dW_t - \frac{\|\psi_t\|^2}{2}dt\right)\right], \tag{9}$$

where $\psi_t - \varphi_t$ is in the kernel of $\sigma(S_t)$ (a.s. for almost all $0 \leq t \leq T$). Hence φ_t is orthogonal to $\psi_t - \varphi_t$, so that $\|\psi_t\| \geq \|\varphi_t\|$. Therefore the estimates (5) and (6) carry over from φ_t to ψ_t.

Proposition 3.1 *Suppose that $(S_t)_{t\geq 0}$ satisfies Assumption 1.2 and has an average squared market price of risk above the threshold $c > 0$; cf. (5).*

For $\varepsilon > 0$ and $0 < \gamma < \frac{c}{2}$ there is $T_0 > 0$ such that, for $T \geq T_0$, condition (ii) of Proposition 2.3 is satisfied with $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = e^{-\gamma T}$, i.e., for each $\mathbf{Q} \in \mathcal{M}_T^e(S)$ there is $A_T \in \mathcal{F}_T$ such that

$$\mathbf{P}[A_T] \leq \varepsilon \quad \textit{and} \quad \mathbf{Q}[A_T] \geq 1 - e^{-\gamma T}.$$

Proof Fix $0 < \gamma < \overline{\gamma} < \frac{c}{2}$ and find $T_0 > \frac{4\overline{\gamma}}{(\overline{\gamma}-\gamma)^2\varepsilon}$ such that, for $T \geq T_0$,

$$\mathbf{P}\left[\frac{1}{T}\int_0^T \|\varphi_t\|^2 dt \leq 2\overline{\gamma}\right] < \frac{\varepsilon}{2}. \tag{10}$$

For $\mathbf{Q}_T \in \mathcal{M}_T^e(S)$ let $(\psi_t)_{0\leq t\leq T}$ be the $\mathbb{R}^N$-valued process as in (9) so that

$$\frac{d\mathbf{Q}_T}{d\mathbf{P}} = Z_T = \exp\left[\int_0^T \left(-\psi_t dW_t - \frac{\|\psi_t\|^2}{2}dt\right)\right].$$

Define the stopping time τ by

$$\tau = \inf\left\{t \in [0, T] \,\middle|\, \int_0^t \|\psi_s\|^2 ds \geq 2\overline{\gamma}T\right\} \wedge T.$$

For the random variable

$$B_\tau = \int_0^\tau \psi_t dW_t$$

we infer from $\int_0^\tau \|\psi_t\|^2 dt \le 2\overline{\gamma}T$ that

$$\|B_\tau\|^2_{L^2(\mathbf{P})} \le 2\overline{\gamma}T,$$

and so Tschebyscheff's inequality implies

$$\mathbf{P}\big[|B_\tau| \ge (\overline{\gamma}-\gamma)T\big] \le \frac{2\overline{\gamma}}{(\overline{\gamma}-\gamma)^2} T^{-1} < \frac{\varepsilon}{2}. \tag{11}$$

For

$$Z_\tau = \exp\left[\int_0^\tau \left(-\psi_t dW_t - \frac{\|\psi_t\|^2}{2} dt\right)\right]$$

we obtain from (10) and (11) that

$$\begin{aligned}
\mathbf{P}\left[Z_\tau > \exp(-\gamma T)\right] &= \mathbf{P}\left[-B_\tau - \int_0^\tau \frac{\|\psi_t\|^2}{2} dt > -\gamma T\right] \\
&\le \mathbf{P}\left[|B_\tau| \ge (\overline{\gamma}-\gamma)T\right] + \mathbf{P}\left[\int_0^\tau \frac{\|\psi_t\|^2}{2} dt < \overline{\gamma}T\right] \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.
\end{aligned}$$

Letting $A_T = \{Z_\tau > \exp(-\gamma T)\}$ we obtain $\mathbf{P}[A_T] \le \varepsilon$ and $\mathbf{Q}_T[A_T^c] = \mathbf{E}[Z_\tau\, A_T^c] \le e^{-\gamma T}$. □

We are now in a position to construct attainable contingent claims which satisfy the arbitrage estimates of Theorem 1.4.

Proof of Theorem 1.4 Let the constant $c > 0$ be given by (5). The preceding Proposition 3.1 implies that, for any given $\varepsilon_1 > 0$ and $\varepsilon_2 = e^{-dT}$, where $0 < d < \frac{c}{2}$, condition (ii) of Proposition 2.3 is satisfied for T sufficiently large. Fix $\varepsilon > 0$ and $\gamma, \gamma_1, \gamma_2$ verifying $0 < \gamma = \gamma_1 + \gamma_2 < \frac{c}{2}$, as in the statement of Theorem 1.4. Fix $d > 0$ with $\gamma < d < \frac{c}{2}$, and let $\mu = \frac{d-\gamma}{d} \in \,]0, 1[$.

We now proceed similarly as in Remark 2.4. Choose T_0 such that for $T > T_0$ condition (ii) of Proposition 2.3 is satisfied with

$$\varepsilon_1 = \left(\frac{\varepsilon}{3}\right)^{\frac{1}{\mu}} \quad \text{and} \quad \varepsilon_2 = e^{-dT}.$$

Defining $\alpha_T \in \,]-\infty, 0[$ by the equation

$$\varepsilon_1 = \varepsilon_2^{|\alpha_T|}.$$

we have that $|\alpha_T| \to 0$ as $T \to \infty$. We may assume that T_0 has been chosen large enough such that $2^{1+|\alpha_T|} \le 3$, for $T \ge T_0$.

Letting $\widetilde{\varepsilon}_1 = 3\varepsilon_1^{\mu} = \varepsilon$ and $\widetilde{\varepsilon}_2 = \varepsilon_2^{(1-\mu)}$ we have

$$\begin{aligned}\widetilde{\varepsilon}_1 \widetilde{\varepsilon}_2^{|\alpha_T|} &= 3\varepsilon_1^{\mu} \varepsilon_2^{(1-\mu)|\alpha_T|} \\ &= 3\varepsilon_1 \ \geq \ 2^{1+|\alpha_T|} \max\left(\varepsilon_1, \varepsilon_2^{|\alpha_T|}\right).\end{aligned}$$

The last inequality implies that condition (iii) of Proposition 2.3 is satisfied with ε replaced by $3\varepsilon_1$. Hence, we may conclude that condition (i) of Proposition 2.3 is satisfied, i.e., there is an $(\widetilde{\varepsilon}_1, \widetilde{\varepsilon}_2)$-arbitrage, for the pair $(\widetilde{\varepsilon}_1, \widetilde{\varepsilon}_2) = \left(\varepsilon, e^{-d(1-\mu)T}\right)$. Note that $d(1-\mu) > \gamma$. Hence, there is $X_T \in K_T$ such that

(i) $X_T \geq -e^{-d(1-\mu)T}$ a.s.
(ii) $\mathbf{P}\left[X_T \geq 1 - e^{-d(1-\mu)T}\right] \geq 1 - \varepsilon$.

Since $d(1-\mu) - \gamma_1 > \gamma_2$, we see that the contingent claim $\overline{X}_T = e^{(d(1-\mu)-\gamma_1)T} X_T \in K_T$ satisfies, for $T \geq T_0$ sufficiently large,

(i) $\overline{X}_T \geq -e^{-\gamma_1 T}$ a.s.,
(ii) $\mathbf{P}[\overline{X}_T \geq e^{\gamma_2 T}] \geq 1 - \varepsilon$.

□

The message of Theorem 1.4 is that we may achieve exponential growth of a portfolio X_T with probability close to 1 as $T \to \infty$, while also controlling the maximal loss by an exponential bound. This implies, for instance, the following lower bound for the growth of logarithmic utility.

Corollary 3.2 *Let $S = (S_t)_{t \geq 0}$ be a process satisfying Assumption 1.2 and having an average squared market price of risk above the threshold $c > 0$ as in (5). For any initial capital $x > 0$, there exist contingent claims $X_T \in K_T$ such that*

$$\liminf_{T\to\infty} \frac{1}{T} \mathbf{E}\left[\log(x + X_T)\right] \geq \frac{c}{2}.$$

Proof By Theorem 1.4 we know that for $\gamma_1 = 0$ and $\gamma_2 < c/2$ we may find, for $\varepsilon > 0$ and T sufficiently large, $X_T \in K_T$ with

(i) $X_T \geq -1$,
(ii) $\mathbf{P}\left[X_T \geq e^{\gamma_2 T}\right] \geq 1 - \varepsilon$.

For $0 < \alpha < x$ we obtain the estimate

$$\frac{1}{T}\mathbf{E}[\log(x + \alpha X_T)] \geq \frac{1}{T}\varepsilon \log(x - \alpha) + \frac{1}{T}(1 - \varepsilon)\log(x + \alpha e^{\gamma_2 T}),$$

hence

$$\frac{1}{T}\mathbf{E}[\log(x + \alpha X_T)] \geq (1 - \varepsilon)\gamma_2,$$

for sufficiently large T, which readily yields the result. □

Theorem 1.4 is valid under the Assumption (5) of a non-trivial market price of risk. If we replace this assumption by the stronger large deviation estimate (6) we expect a stronger result: the term $\mathbf{P}[X_T < e^{\gamma_2 T}]$ in statement (ii) of Theorem 1.4 should even decay exponentially as $T \to \infty$. Let us sketch a systematic approach how to obtain such a sharper result; we also take the opportunity to motivate the large deviation estimate (6).

Let $(S_t)_{t\geq 0}$ be a d-dimensional diffusion satisfying Assumption 1.2, and assume that S is in fact ergodic. Thus, the empirical distributions

$$\rho_T(\omega) := \frac{1}{T}\int_0^T \delta_{S_t(\omega)}dt$$

converge weakly, $\mathbf{P}$-a.s., to the unique invariant distribution μ of S. Typically, the empirical distributions will satisfy a large deviations principle of the form

$$\frac{1}{T}\log \mathbf{P}[\rho_T \in A] \asymp -\inf_{\nu\in A} I(\nu) \tag{12}$$

with some rate function I defined on the convex set $\mathcal{M}_1(\mathbb{R}^d)$ of probability measures on $\mathbb{R}^d$, where (12) should be read as an upper bound for the lim sup of the left-hand side if A is a closed subset of $\mathcal{M}_1(\mathbb{R}^d)$ and as a lower bound for the lim inf if A is open. Under some regularity conditions the market price of risk will then satisfy the large deviations estimate (6)

$$\lim_{T\nearrow\infty}\frac{1}{T}\log \mathbf{P}\left[\frac{1}{T}\int_0^T \|\varphi_t\|^2 dt \leq c_1\right] \leq -c_2$$

for any $c_1 < \int \|\varphi(x)\|^2\mu(dx)$, with

$$c_2 := \inf\left\{ I(\nu) \,\middle|\, \int_0^T \|\varphi(x)\|^2 d\nu(x) \leq c_1 \right\} > 0,$$

due to the contraction principle (see, e.g., [8]).

In such a situation one should expect an exponential decay of the probabilities

$$\mathbf{P}\left[Z_T \geq e^{-\gamma T}\right]$$

and a corresponding exponential version of the estimates in Theorem 1.4, assuming now uniqueness of the locally equivalent martingale measure $\mathbf{Q}$. However, this will involve a slight refinement of the large deviations principle in its classical form (12).

Let us sketch the argument for the one-dimensional case. Indeed,

$$\mathbf{P}\left[Z_T \geq e^{-\gamma T}\right] = \mathbf{P}\left[\int_0^T \varphi_t dW_t + \frac{1}{2}\int_0^T \varphi_t^2 dt \leq \gamma T\right].$$

By Itô's formula,

$$\begin{aligned}
\int_0^T \varphi_t dW_t &:= \int_0^T \varphi_t \left(\frac{dS_t}{\sigma(S_t)} - \varphi_t dt\right) \\
&= \int_0^T f(S_t)dS_t - \int_0^T \varphi_t^2 dt \\
&= F(S_T) - F(S_0) - \frac{1}{2}\int_0^T f'(S_t)\sigma^2(S_t)dt - \int_0^T \varphi^2(S_t)dt,
\end{aligned}$$

where we put $f(x) := \frac{\varphi(x)}{\sigma(x)}$, assume $f \in C^1$, and take $F \in C^2$ such that $F' = f$. Thus,

$$\mathbf{P}\left[Z_T \geq e^{-\gamma T}\right] = \mathbf{P}\left[(F(S_T) - F(S_0)) - \frac{1}{2}\int_0^T h(S_t)dt \leq \gamma T\right]$$
$$= \mathbf{P}\left[\frac{1}{T}(F(S_T) - F(S_0)) - \frac{1}{2}\int_0^T h(x)d\rho_T(x) \leq \gamma\right]$$

where we define $h(x) = f'(x)\sigma^2(x) + \varphi^2(x)$. An exponential estimate of the form

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbf{P}\left[Z_T \geq e^{-\gamma T}\right] < 0$$

will now follow for $\gamma < -\frac{1}{2}\int h d\mu$, if we have a joint large deviation principle for the random variables $\left(\frac{1}{T}\left(F(S_T) - F(S_0)\right), \rho_T\right)$ with values in $\mathbb{R}^1 \times \mathcal{M}_1(\mathbb{R}^d)$ or, more specifically, for the random variables

$$\frac{1}{T}(F(S_T) - F(S_0)) - \frac{1}{2}\int_0^T h(x)d\rho_T(x).$$

In this paper, we do not try to give a rigorous version of the argument under general conditions. Instead we will illustrate this approach by proving explicit exponential estimates for the geometric Ornstein–Uhlenbeck process.

4 A case study: the geometric Ornstein–Uhlenbeck process

In this section we consider the geometric Ornstein–Uhlenbeck process and derive exponential versions of the extimates in Theorem 1.4. In order to keep the notation as transparent as possible, we take the simplest case

$$S_t = \exp(Y_t), \tag{13}$$

where $(Y_t)_{t\geq 0}$ is the stationary Ornstein–Uhlenbeck process defined by

$$dY_t = -\rho Y_t dt + \sigma dW_t, \quad Y_0 = y_0,$$

with parameters $\rho > 0$ and $\sigma > 0$ and with initial value $y_0 \in \mathbb{R}$; see, however, Remark 4.4 below for the extension to an Ornstein–Uhlenbeck process with drift.

The process $S = (S_t)_{t\geq 0}$ defined by (13) satisfies the SDE

$$dS_t = S_t[-\rho Y_t dt + \sigma dW_t] + S_t\frac{\sigma^2}{2}dt$$
$$= S_t\sigma\left[dW_t - \frac{1}{\sigma}\left(\rho Y_t - \frac{\sigma^2}{2}\right)dt\right]. \tag{14}$$

For fixed $T > 0$, the unique equivalent martingale measure $\mathbf{Q}$ on $\mathcal{F}_T$ for the process S is defined by

$$\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathcal{F}_T} = Z_T$$

where

$$Z_T := \exp\left(\int_0^T \frac{1}{\sigma}\left(\rho Y_t - \frac{\sigma^2}{2}\right) dW_t - \frac{1}{2}\int_0^T \frac{1}{\sigma^2}\left(\rho Y_t - \frac{\sigma^2}{2}\right)^2 dt\right). \tag{15}$$

In other words, the dynamics of S under Q takes the form

$$dS_t = \sigma S_t dW_t^{\mathbf{Q}},$$

where

$$\begin{aligned} W_t^{\mathbf{Q}} &:= W_t - \int_0^t \frac{1}{\sigma}\left(\rho Y_s - \frac{\sigma^2}{2}\right) ds \\ &= \frac{1}{\sigma}\left(Y_t - y_0 + \frac{\sigma^2}{2}t\right) \end{aligned} \tag{16}$$

is a Wiener process under $\mathbf{Q}$.

In this specific model we can describe the large deviations more explicitly.

Proposition 4.1 *For any* $\gamma \in \left]0, \frac{\sigma^2}{8} + \frac{\rho}{4}\right[$, *the sets* $A_T := \left\{Z_T \geq e^{-\gamma T}\right\}$ *satisfy*

$$\mathbf{Q}[A_T] \geq 1 - e^{-\gamma T} \tag{17}$$

and

$$\lim_{T \nearrow \infty} \frac{1}{T} \log \mathbf{P}[A_T] = -\frac{\left(\frac{\sigma^2}{8} - \gamma + \frac{\rho}{4}\right)^2}{\frac{\sigma^2}{8} - \gamma + \frac{\rho}{2}}. \tag{18}$$

Proof Clearly,

$$\mathbf{Q}\left[Z_T \leq e^{-\gamma T}\right] = \int_{\{Z_T \leq e^{-\gamma T}\}} Z_T d\mathbf{P} \leq e^{-\gamma T},$$

and this implies (17). The proof of (18) consists in combining a result of Florens–Landais and Pham [14] for certain large deviations of the Ornstein–Uhlenbeck process with a simple perturbation argument. To this end, we write

$$\begin{aligned} \log Z_T &= \frac{1}{\sigma}\int_0^T \left(\rho Y_t - \frac{\sigma^2}{2}\right)\left(\frac{1}{\sigma} dY_t + \frac{\rho}{\sigma} Y_t dt\right) - \frac{1}{2}\frac{\rho^2}{\sigma^2}\int_0^T Y_t^2 dt - \frac{\sigma^2}{8}T + \frac{\rho}{2}\int_0^T Y_t dt \\ &= \frac{\rho}{\sigma^2}\int_0^T Y_t dY_t - \frac{1}{2}(Y_T - Y_0) + \frac{\rho^2}{2\sigma^2}\int_0^T Y_t^2 dt - \frac{\sigma^2}{8}T \\ &= \frac{\rho}{\sigma^2}\xi_T - \frac{\sigma^2}{8}T, \end{aligned} \tag{19}$$

where

$$\begin{aligned}
\xi_T &:= \eta_T + \zeta_T, \\
\eta_T &:= \int_0^T Y_t dY_t + \frac{\rho}{2}\int_0^T Y_t^2 dt \\
&= \frac{1}{2}(Y_T^2 - Y_0^2 - \sigma^2 T) + \frac{\rho}{2}\int_0^T Y_t^2 dt, \\
\zeta_T &:= \frac{\sigma^2}{2\rho}(Y_0 - Y_T).
\end{aligned}$$

Recall that the Ornstein–Uhlenbeck process Y is ergodic with invariant distribution $N(0, \frac{\sigma^2}{2\rho})$, and note that this implies

$$\lim_{T\nearrow\infty}\frac{\xi_T}{T} = \lim_{T\nearrow\infty}\frac{\eta_T}{T} = -\frac{\sigma^2}{2} + \frac{\rho}{2}\frac{\sigma^2}{2\rho} = -\frac{\sigma^2}{4} \quad \mathbf{P}\text{-a.s.} \tag{20}$$

Let us now look at the corresponding large deviations. Applying Theorem 2.2 of Florens–Landais and Pham [14] (with $\theta_0 = -\rho$, $\theta = -\frac{\rho}{2}$, and the straightforward extension to the case $\sigma \neq 1$), we see that the random variables $\frac{\eta_T}{T}$ satisfy a large deviations principle with rate function

$$I(y) = \begin{cases} \frac{2\rho\left(\frac{y}{\sigma^2}+\frac{1}{4}\right)^2}{2\frac{y}{\sigma^2}+1} & \text{for } y > -\frac{1}{2}\sigma^2 \\ \infty & \text{for } y \le -\frac{1}{2}\sigma^2 \end{cases}$$

i.e.,

$$\limsup_{T\nearrow\infty}\frac{1}{T}\log\mathbf{P}\left[\frac{\eta_T}{T}\in F\right] \le -\inf_{y\in F} I(y)$$

for any closed set $F \subseteq \mathbb{R}^1$ and

$$\liminf_{T\nearrow\infty}\frac{1}{T}\log\mathbf{P}\left[\frac{\eta_T}{T}\in G\right] \ge -\inf_{y\in G} I(y)$$

for any open set $G \subseteq \mathbb{R}^1$. A simple perturbation argument shows that the same large deviations principle with rate function I applies to the random variables $\frac{\xi_T}{T}$. Indeed, the additional terms ζ_T are normally distributed with means m_T converging to $m_\infty \in \mathbb{R}^1$ and variances σ_T^2 converging to $\sigma_\infty^2 \in]0, \infty[$, and this implies

$$\begin{aligned}
\lim_{T\nearrow\infty}\frac{1}{T^2}\log\mathbf{P}\left[\frac{|\zeta_T|}{T} > \varepsilon\right] &= \lim_{T\nearrow\infty}\frac{1}{T^2}\log\frac{2}{\sqrt{2\pi}}\int_{\varepsilon T\sigma_T^{-1}}^{\infty} e^{-\frac{y^2}{2}}dy \\
&= -\frac{\varepsilon^2}{2\sigma_\infty^2}.
\end{aligned}$$

In particular,

$$\mathbf{P}\left[\frac{\xi_T}{T} \in F\right] \leq \mathbf{P}\left[\frac{|\xi_T|}{T} > \varepsilon\right] + \mathbf{P}\left[\frac{\eta_T}{T} \in F_\varepsilon\right]$$
$$\leq 2\mathbf{P}\left[\frac{\eta_T}{T} \in F_\varepsilon\right]$$

for any $\varepsilon > 0$, $F_\varepsilon := \{y \mid d(y, F) \leq \varepsilon\}$, and for T large enough. Thus,

$$\overline{\lim_{T \nearrow \infty}} \frac{1}{T} \log \mathbf{P}\left[\frac{\xi_T}{T} \in F\right] \leq - \inf_{y \in F_\varepsilon} I(y)$$

for closed F and for arbitrary $\varepsilon > 0$, hence

$$\overline{\lim_{T \nearrow \infty}} \frac{1}{T} \log \mathbf{P}\left[\frac{\xi_T}{T} \in F\right] \leq - \inf_{y \in F} I(y).$$

The lower bound for open sets follows in the same manner. For $y_0 > -\frac{\sigma^2}{4}$ and $F := [y_0, \infty[$, these two bounds imply

$$\lim_{T \nearrow \infty} \frac{1}{T} \log \mathbf{P}\left[\frac{\xi_T}{T} \geq y_0\right] = -I(y_0).$$

We can now conclude that the asymptotic behavior of

$$\mathbf{P}\left[Z_T \geq e^{-\gamma T}\right] = \mathbf{P}\left[\frac{\rho}{\sigma^2}\xi_T - \frac{1}{8}\sigma^2 T \geq -\gamma T\right]$$
$$= \mathbf{P}\left[\frac{\xi_T}{T} \geq \frac{\sigma^2}{\rho}\left(\frac{\sigma^2}{8} - \gamma\right)\right]$$

for $\gamma \in \left]0, \frac{\sigma^2}{8} + \frac{\rho}{4}\right[$ is described by Eq. (18). □

We now are prepared to prove Theorem 1.5. In fact we will prove the following stronger version:

Theorem 4.2 *Let $(S_t)_{t \geq 0}$ be the geometric Ornstein–Uhlenbeck process defined in* (7). *Take $\gamma \in \left]0, \frac{\sigma^2}{8} + \frac{\rho}{4}\right[$ and any $\gamma_1 < \gamma$. Then there exist contingent claims $X_T \in K_T$ such that, for any $\gamma_2 < \gamma - \gamma_1$,*

(i) $X_T \geq -e^{-\gamma_2 T}$ *for large T,*

(ii) $\lim_{T \to \infty} \frac{1}{T} \log \mathbf{P}\left[X_T < e^{\gamma_1 T}\right] = \frac{-\left(\frac{\sigma^2}{8} - \gamma + \frac{\rho}{4}\right)^2}{\frac{\sigma^2}{8} - \gamma + \frac{\rho}{2}}.$

More precisely, we can achieve the tighter bounds

(iii) $X_T \geq -\alpha_T := -e^{\gamma_1 T} \frac{\mathbf{Q}[A_T^c]}{\mathbf{Q}[A_T]}$

where $A_T := \{Z_T \geq e^{-\gamma T}\}$, and the decay rate in (ii) for the shortfall probabilities $\mathbf{P}[X_T < e^{\gamma_1 T}]$ is in fact optimal under the constraint (iii).

Proof Since $\mathbf{Q}[A_T^c] \leq e^{-\gamma T}$ by (17), the constants α_T defined by (iii) satisfy

$$\alpha_T \leq e^{-(\gamma - \gamma_1)T} \frac{1}{\mathbf{Q}[A_T]},$$

hence $\alpha_T \le e^{-\gamma_2 T}$ for large T if $\gamma_2 < \gamma - \gamma_1$. Now consider the contingent claims

$$X_T := e^{\gamma_1 T}\mathbf{1}_{A_T^c} - \alpha_T \mathbf{1}_{A_T}.$$

Clearly, $X_T \ge -\alpha_T$, and

$$\mathbf{E_Q}[X_T] = e^{\gamma_1 T}\mathbf{Q}[A_T^c] - \alpha_T \mathbf{Q}[A_T] = 0,$$

hence $X_T \in K_T$. Moreover, since

$$\left\{X_T < e^{\gamma_1 T}\right\} = A_T$$

for large enough T, part (ii) of Theorem 4.2 follows from (18). In order to check the optimality of the convergence rate in (ii) under the constraint (iii), take a sequence of contingent claims $\widetilde{X}_T \in K_T$ such that $\widetilde{X}_T \ge -\alpha_T$. The corresponding sets $\widetilde{A}_T := \left\{\widetilde{X}_T < e^{\gamma_1 T}\right\}$ satisfy

$$\begin{aligned}\mathbf{Q}\left[\widetilde{A}_T^c\right]\left(e^{\gamma_1 T} + \alpha_T\right) - \alpha_T &= e^{\gamma_1 T}\mathbf{Q}\left[\widetilde{A}_T^c\right] + (-\alpha_T)\mathbf{Q}\left[\widetilde{A}_T\right] \\ &\le \mathbf{E_Q}\left[\widetilde{X}_T\right] \le 0,\end{aligned}$$

hence

$$\mathbf{Q}\left[\widetilde{A}_T^c\right] \le \frac{\alpha_T}{\alpha_T + e^{\gamma_1 T}} = \mathbf{Q}[A_T^c].$$

The Neyman–Pearson lemma allows us to conclude that

$$\mathbf{P}\left[\widetilde{A}_T^c\right] \le \mathbf{P}[A_T^c].$$

In particular, the shortfall probabilities $\mathbf{P}\left[\widetilde{X}_T < e^{\gamma_1 T}\right] = \mathbf{P}[\widetilde{A}_T]$ cannot decay at a faster rate than described by (ii). □

Remark 4.3 Note that the constants α_T defined in part (iii) of Theorem 4.2 satisfy

$$\begin{aligned}\overline{\lim_{t\to\infty}} \frac{1}{T}\log \alpha_T &= \gamma_1 + \overline{\lim_{T\to\infty}} \frac{1}{T}\log \mathbf{Q}[A_T^c] \\ &\le -(\gamma - \gamma_1),\end{aligned} \tag{21}$$

where we have used the crude estimate $\mathbf{Q}[A_T^c] \le e^{-\gamma T}$. Refined large deviation estimates for the probabilities $\mathbf{Q}[A_T^c]$ yield better rates for the convergence of α_T to 0. Indeed,

$$\begin{aligned}\mathbf{Q}[A_T^c] &= \mathbf{Q}[Z_T < e^{-\gamma T}] \\ &\le e^{-\eta\gamma T}\mathbf{E_Q}[Z_T^{-\eta}] \\ &= e^{-\eta\gamma T}\mathbf{E_P}[Z_T^{1-\eta}]\end{aligned}$$

for any $\eta > 0$. The term $\mathbf{E_P}[Z_T^{1-\eta}]$ can be computed explicitly; cf. Proposition 5.6. This yields

$$\overline{\lim_{T\to\infty}} \frac{1}{T}\log \mathbf{Q}[A_T^c] \le -(\gamma + f(\eta))$$

with

$$f(\eta) := \frac{\rho}{2}(\sqrt{\eta} - \eta) + (1-\eta)\left(\frac{\sigma^2}{8} - \gamma\right).$$

Since we have assumed $\gamma < \frac{\sigma^2}{8} + \frac{\rho}{4}$, the function f attains its maximum value

$$f(\eta(\gamma)) = \left(\frac{\sigma^2}{8} + \frac{\rho}{4} - \gamma\right)^2 \left(\frac{\sigma^2}{8} + \frac{\rho}{2} - \gamma\right)^{-1} > 0$$

in $\eta(\gamma) = \frac{\rho^2}{16}\left(\frac{\sigma^2}{8} + \frac{\rho}{2} - \gamma\right)^{-2}$, and so we can replace the rate $\gamma - \gamma_1$ in (21) by the better rate $\gamma - \gamma_1 + f(\eta(\gamma))$.

Remark 4.4 The preceding discussion also applies to more general versions of the geometric Ornstein–Uhlenbeck process. Suppose, for example, that the discounted price process is of the form

$$S_t = e^{Y_t + \mu t}$$

with some constant drift parameter μ. In this case, the density Z_T of the unique equivalent martingale measure is again of the form (15), but with modified parameters

$$\widetilde{\sigma} = \sigma + \frac{2\mu}{\sigma}, \quad \widetilde{\rho} = \rho\left(1 + \frac{2\mu}{\sigma^2}\right).$$

As a result, Eq. (18) in Proposition 4.1 is still valid, but with $\widetilde{\sigma}$ instead of σ. Indeed, note that $\log Z_T$ takes the form (19) with $\widetilde{\xi}_T = \eta_T + \widetilde{\zeta}_T$, where

$$\widetilde{\zeta}_T = \frac{\sigma\widetilde{\sigma}}{2\rho}(Y_0 - Y_T).$$

The law of large numbers (20) remains valid for $\widetilde{\xi}$, and the large deviations principle for η can be transferred also to $\widetilde{\xi}$. Thus

$$\begin{aligned}\lim_{T\nearrow\infty} \frac{1}{T}\log \mathbf{P}\left[Z_T \geq e^{-\gamma T}\right] &= \lim_{T\nearrow\infty} \frac{1}{T}\log \mathbf{P}\left[\frac{\eta_T}{T} \geq \frac{\sigma^2}{\rho}\left(\frac{1}{8}\widetilde{\sigma}^2 - \gamma\right)\right] \\ &= -I\left(\frac{\sigma^2}{\rho}\left(\frac{1}{8}\widetilde{\sigma}^2 - \gamma\right)\right)\end{aligned}$$

for $\gamma \in \left]0, \frac{\widetilde{\sigma}^2}{8} + \frac{\rho}{4}\right[$, and this amounts to Eq. (18) with $\widetilde{\sigma}$ instead of σ.

Finally we want to compare the arbitrage estimates for the geometric Ornstein–Uhlenbeck case as described in Theorem 1.5 and in Remark 4.4 with the much simpler case of the Black–Scholes model.

Theorem 4.5 *Let $\mu \neq 0$, $\sigma > 0$ and define the Black–Scholes model by*

$$S_t = S_0 \exp\left[\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right)t\right].$$

Take $\gamma \in \left]0, \frac{\varphi^2}{2}\right[$, where $\varphi_t \equiv \varphi \equiv \frac{\mu}{\sigma}$. For any $\gamma_1 < \gamma$, there exists $X_T \in K_T$ such that, for any $\gamma_2 < \gamma - \gamma_1$,

(i) $X_T \geq -e^{-\gamma_2 T}$ *for large T,*

(ii) $\lim_{T\to\infty} \frac{1}{T}\log \mathbf{P}\left[X_T < e^{\gamma_1 T}\right] = -\frac{1}{2}\left(\frac{\varphi}{2} - \frac{\gamma}{\varphi}\right)^2$.

More precisely, we can achieve the tighter bounds

(iii) $X_T \geq -\alpha_T := -e^{\gamma_1 T}\frac{\mathbf{Q}[A_T^c]}{\mathbf{Q}[A_T]}$, *where* $A_T = \{Z_T \geq e^{-\gamma T}\}$. *Then*

$$\lim_{T\to\infty} \frac{1}{T} \log \alpha_T = \gamma_1 - \left(\frac{\gamma}{\varphi} + \frac{\varphi}{2}\right)^2,$$

and the decay rate for the shortfall probabilities in (ii) is optimal under the constraint (iii).

Proof For $T > 0$ the density Z_T of the unique equivalent martingale measure is given by

$$Z_T = \exp\left[-\varphi W_T - \frac{\varphi^2}{2} T\right].$$

We have

$$\begin{aligned}\mathbf{P}\left[A_T^c\right] &= \mathbf{P}\left[-\varphi W_T - \frac{\varphi^2}{2} T < -\gamma T\right] \\ &= \mathbf{P}\left[-W_T < \frac{\left(\frac{\varphi^2}{2} - \gamma\right) T}{\varphi}\right] \\ &= \Phi\left(\left(\frac{\varphi}{2} - \frac{\gamma}{\varphi}\right) T^{\frac{1}{2}}\right)\end{aligned}$$

for $\varphi > 0$, and similarly for $\varphi < 0$, where Φ denotes the distribution function of a standard normal variable. On the other hand, since $W_t^{\mathbf{Q}} := W_t + \varphi t$ defines a Wiener process under $\mathbf{Q}$,

$$\begin{aligned}\mathbf{Q}\left[A_T^c\right] &= \mathbf{Q}\left[-\varphi W_T^{\mathbf{Q}} < -\gamma T - \frac{\varphi}{2} T\right] \\ &= \Phi\left(-\left(\frac{\gamma}{\varphi} + \frac{\varphi}{2}\right)\sqrt{T}\right).\end{aligned}$$

Now consider the contingent claim

$$X_T := e^{\gamma_1 T} \mathbf{1}_{A_T^c} - \alpha_T \mathbf{1}_{A_T}.$$

Clearly, $X_T \geq -\alpha_T$ and $\mathbf{E}_{\mathbf{Q}}[X_T] = 0$ hence $X_T \in K_T$. Finally we have

$$\mathbf{P}\left[X_T < e^{\gamma_2 T}\right] = \mathbf{P}\left[A_T\right] = \Phi\left(-\left(\frac{\varphi}{2} - \frac{\gamma}{\varphi}\right) T^{\frac{1}{2}}\right).$$

Since

$$\lim_{T\to\infty} \frac{1}{T} \log \Phi\left(-\eta T^{\frac{1}{2}}\right) = -\frac{\eta^2}{2}$$

for any $\eta \in \mathbb{R}^1$, we easily obtain

$$\lim_{T\to\infty} \frac{1}{T} \log \mathbf{P}\left[X_T < e^{\gamma_1 T}\right] = -\left(\frac{\varphi}{2} - \frac{\gamma}{\varphi}\right)^2 /2.$$

In the same way we see that

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T} \log \alpha_T &= \gamma_1 + \lim_{T\to\infty} \frac{1}{T} \log \mathbf{Q}[A_T^c] \\ &= \gamma_1 - \left(\frac{\gamma}{\varphi} + \frac{\varphi}{2}\right)^2.\end{aligned}$$

As in the proof of Theorem 4.2, optimality of the decay rate in (ii) under the constraints (iii) follows by applying the Neyman–Pearson lemma. □

5 Optimal strategies for the geometric Ornstein–Uhlenbeck process for HARA utilities

Let us return to the geometric Ornstein–Uhlenbeck model introduced in Sect. 4. For three standard choices of a utility function U, we are going to compute the maximal expected utility

$$u_T(x) := \mathbf{E}_{\mathbf{P}}[U(X_T)]$$

attainable at time $T > 0$ using some self-financing trading strategy and the initial capital $x > 0$. Recall (see, e.g, [20,21]) that the optimal contingent claim X_T is of the form

$$X_T = (U')^{-1}(yZ_T) \tag{22}$$

where the Lagrange multiplier $y > 0$ is given by

$$\mathbf{E}_{\mathbf{Q}}\left[(U')^{-1}(yZ_T)\right] = x. \tag{23}$$

Hence $X_T - x \in K_T$. A little warning on the notation seems appropriate: While in the previous sections it was natural to have $X_T \in K_T$, we now follow the usual notation in utility maximization, where we fix an initial endowment $x \in \mathbb{R}$ and let X_T denote a random variable such that $X_T - x \in K_T$.

We are also going to identify the optimal trading strategy, i.e., the predictable process $(\xi_t^{(T)})_{0\leq t\leq T}$ such that

$$\begin{aligned} X_t^{(T)} &:= \mathbf{E}_{\mathbf{Q}}[X_T \mid \mathcal{F}_t] \\ &= x + \int_0^t \xi_s^{(T)} dS_s \end{aligned} \tag{24}$$

for any $t \in [0, T]$. It is clearly equivalent to compute the optimal proportion

$$\pi_t^{(T)} := \frac{\xi_t^{(T)} \cdot S_t}{X_t^{(T)}} \tag{25}$$

of the capital $X_t^{(T)}$ generated up to time t which should be invested in the financial asset.

Moreover, we are going to describe the growth of $u_T(x)$ and the limiting form of the optimal strategy as T increases to ∞, and to give some financial interpretation in terms of certainty equivalents.

5.1 Logarithmic utility

For the logarithmic utility function $U(x) = \log x$ it follows from (22) and (23) that the optimal contingent claim at time T is given by

$$X_T = xZ_T^{-1}. \tag{26}$$

Thus the maximal expected utility takes the form

$$u_T(x) = \log x + H_T(\mathbf{P}|\mathbf{Q}) \tag{27}$$

where

$$H_T(\mathbf{P}|\mathbf{Q}) := \mathbf{E}_{\mathbf{P}}\left[\log \frac{d\mathbf{P}}{d\mathbf{Q}}\bigg|_{\mathcal{F}_T}\right] = \mathbf{E}_{\mathbf{P}}\left[\log Z_T^{-1}\right]$$

denotes the relative entropy of $\mathbf{P}$ with respect to $\mathbf{Q}$ on $\mathcal{F}_T$.

Proposition 5.1 *The maximal expected utility at time T is given by*

$$u_T(x) = \log x + \frac{1}{4}\rho T - \frac{1}{8}\left(1 - e^{-2\rho T}\right) + \frac{1}{4}\frac{\rho}{\sigma^2}y_0^2\left(1 - e^{-2\rho T}\right) - \frac{1}{2}y_0\left(1 - e^{-\rho T}\right) + \frac{1}{8}\sigma^2 T. \tag{28}$$

In particular $u_T(x)$ grows linearly at the rate

$$\lim_{T\nearrow\infty}\frac{u_T(x)}{T} = \frac{1}{4}\rho + \frac{1}{8}\sigma^2. \tag{29}$$

Proof In view of (27) it is enough to compute the relative entropy

$$H_T(\mathbf{P}|\mathbf{Q}) = -\mathbf{E}_{\mathbf{P}}\left[\int_0^T \frac{1}{\sigma}\left(\rho Y_t - \frac{1}{2}\sigma^2\right)dW_t\right] + \mathbf{E}_{\mathbf{P}}\left[\frac{1}{2}\int_0^T \frac{1}{\sigma^2}\left(\rho Y_t - \frac{1}{2}\sigma^2\right)^2 dt\right].$$

Note that the first term vanishes, and recall that the first two moments of the Ornstein–Uhlenbeck process $(Y_t)_{t\geq 0}$ are given by

$$\mathbf{E}_{\mathbf{P}}[Y_t] = y_0 e^{-\rho t} \tag{30}$$

and

$$\mathbf{E}_{\mathbf{P}}[Y_t^2] = \frac{\sigma^2}{2\rho}(1 - e^{-2\rho t}) + e^{-2\rho t}y_0^2.$$

Thus

$$\begin{aligned} H_T(\mathbf{P}|\mathbf{Q}) &= \frac{\rho^2}{2\sigma^2}\int_0^T \mathbf{E}_{\mathbf{P}}[Y_t^2]dt - \frac{\rho}{2}\int_0^T \mathbf{E}_{\mathbf{P}}[Y_t]dt + \frac{1}{8}\sigma^2 T \\ &= \frac{1}{4}\frac{\rho}{\sigma^2}y_0^2\left(1 - e^{-2\rho T}\right) - \frac{1}{2}y_0\left(1 - e^{-\rho T}\right) - \frac{1}{8}\left(1 - e^{-2\rho T}\right) + \frac{1}{4}\rho T + \frac{1}{8}\sigma^2 T. \end{aligned}$$

In the logarithmic case the optimal strategy does not depend on the horizon T:

Proposition 5.2 *The optimal proportion defined by (25) is given by*

$$\pi_t^{(T)} = \frac{1}{2} - \frac{1}{\sigma^2}\rho Y_t. \tag{31}$$

Proof In view of (15), (16) and (26) the optimal contingent claim is given by

$$\begin{aligned} X_T &= x\exp\left(-\int_0^T \frac{1}{\sigma}\left(\rho Y_t - \frac{1}{2}\sigma^2\right)dW_t + \frac{1}{2}\int_0^T \frac{1}{\sigma^2}\left(\rho Y_t - \frac{1}{2}\sigma^2\right)^2 dt\right) \\ &= x\exp\left(-\int_0^T \frac{1}{\sigma}\left(\rho Y_t - \frac{1}{2}\sigma^2\right)dW_t^{\mathbf{Q}} - \frac{1}{2}\int_0^T \frac{1}{\sigma^2}\left(\rho Y_t - \frac{1}{2}\sigma^2\right)^2 dt\right). \end{aligned}$$

Thus the $\mathbf{Q}$-martingale

$$X_t^{(T)} := \mathbf{E}_{\mathbf{Q}}[X_T \mid \mathcal{F}_t]$$
$$= x \exp\left(-\int_0^t \frac{1}{\sigma}\left(\rho Y_s - \frac{1}{2}\sigma^2\right) dW_s^{\mathbf{Q}} - \frac{1}{2}\int_0^t \frac{1}{\sigma^2}\left(\rho Y_s - \frac{1}{2}\sigma^2\right)^2 ds\right)$$

satisfies

$$dX_t^{(T)} = X_t^{(T)}\left(-\frac{1}{\sigma}\left(\rho Y_t - \frac{1}{2}\sigma^2\right)\right) dW_t^{\mathbf{Q}}$$
$$= X_t^{(T)}\left(-\frac{1}{\sigma^2}\rho Y_t + \frac{1}{2}\right) S_t^{-1} dS_t,$$

and so the proportion $\pi_t^{(T)}$ defined by (25) is indeed given by (31). □

Remark 5.3 It is well-known that for logarithmic utility the optimal contingent claim is given by the "numéraire portfolio"

$$X_T = x\left(Z_T^{\min}\right)^{-1} = x + \int_0^T X_t\left(\frac{\varphi}{\sigma}\right)_t dS_t, \tag{32}$$

i.e., the optimal strategy is simply proportional to the current wealth X_t as well as to the quotient $\frac{\varphi}{\sigma}$; see, e.g., [2] and [24]. Note that these results also hold true in the general incomplete case. In our special case of the geometric Ornstein–Uhlenbeck process, this is of course consistent with the explicit formula (31) for the optimal proportion. Formula (32) implies in particular that any estimate (from below) on $-\mathbf{E}\left[\log\left(Z^{\min}\right)\right]$ yields an estimate (from below) on $\mathbf{E}[\log(X_T)]$.

5.2 Exponential utility

For the exponential utility function $U(x) = -\exp(-\lambda x)$ with parameter $\lambda > 0$ we have $(U')^{-1}(y) = \frac{1}{\lambda}\log(\frac{\lambda}{y})$, so that the optimal contingent claim in (22) takes the form

$$X_T = x + \frac{1}{\lambda}\left(H_T(\mathbf{Q}|\mathbf{P}) - \log Z_T\right),$$

where

$$H_T(\mathbf{Q}|\mathbf{P}) := \mathbf{E}_{\mathbf{Q}}[\log Z_T]$$

denotes the relative entropy of $\mathbf{Q}$ with respect to $\mathbf{P}$ on $\mathcal{F}_T$.

Proposition 5.4 *The maximal expected utility is given by*

$$u_T(x) = -\exp\left(-\lambda x - H_T(\mathbf{Q}|\mathbf{P})\right) \tag{33}$$

where

$$H_T(\mathbf{Q}|\mathbf{P}) = \left(\frac{1}{2}\frac{\rho^2}{\sigma^2}y_0^2 - \frac{\rho}{2}y_0 + \frac{1}{8}\sigma^2\right)T + \left(\frac{1}{4}\rho^2(1-y_0) + \frac{1}{8}\rho\sigma^2\right)T^2 + \frac{1}{24}\rho^2\sigma^2 T^3. \tag{34}$$

In particular, $u_T(x)$ grows to its upper bound 0 at the rate

$$\lim_{T\nearrow\infty}\frac{1}{T^3}\log\left(-u_T(x)\right) = -\frac{1}{24}\rho^2\sigma^2.$$

Proof Since by (22)

$$\begin{aligned} u_T(x) &= -\mathbf{E}_{\mathbf{P}}\left[\exp(-\lambda X_T)\right] \\ &= -\exp\left(-\lambda x - H_T(\mathbf{Q}|\mathbf{P})\right), \end{aligned}$$

it remains to compute the relative entropy in our special setting. In view of (15) and (16) we have

$$\begin{aligned} H_T(\mathbf{Q}|\mathbf{P}) &= \mathbf{E}_{\mathbf{Q}}\left[\int_0^T \frac{1}{\sigma}\left(\rho Y_t - \frac{1}{2}\sigma^2\right) dW_t - \frac{1}{2}\int_0^T \frac{1}{\sigma^2}\left(\rho Y_t - \frac{1}{2}\sigma^2\right)^2 dt\right] \\ &= \mathbf{E}_{\mathbf{Q}}\left[\int_0^T \frac{1}{\sigma}\left(\rho Y_t - \frac{1}{2}\sigma^2\right) dW_t^{\mathbf{Q}} + \frac{1}{2}\int_0^T \frac{1}{\sigma^2}\left(\rho Y_t - \frac{1}{2}\sigma^2\right)^2 ds\right] \\ &= \frac{\rho^2}{2\sigma^2}\int_0^T \mathbf{E}_{\mathbf{Q}}[Y_t^2]dt - \frac{\rho}{2}\int_0^T \mathbf{E}_{\mathbf{Q}}[Y_t]dt + \frac{1}{8}\sigma^2 T. \end{aligned}$$

Since $Y_t = \sigma W_t^{\mathbf{Q}} + y_0 - \frac{1}{2}\sigma^2 t$ by (16), the first two moments are given by

$$\mathbf{E}_{\mathbf{Q}}[Y_t] = y_0 - \frac{1}{2}\sigma^2 t.$$

and

$$\mathbf{E}_{\mathbf{Q}}[Y_t^2] = \sigma^2 t + y_0^2 - y_0\sigma^2 t + \frac{1}{4}\sigma^4 t^2,$$

and this yields Eq. (34). □

Let us now identify the optimal strategy for a fixed horizon $T > 0$.

Proposition 5.5 *The optimal quantity* $\xi_t^{(T)}$ *defined by* (24) *is given by*

$$\xi_t^{(T)} = \frac{1}{\lambda S_t}\left[-\frac{1}{\sigma^2}\left(\rho + \rho^2(T-t)\right) Y_t + \frac{1}{4}\left(1 + \rho(T-t)\right)^2 + \frac{1}{4}\right]. \tag{35}$$

Proof Consider the **Q**-martingale

$$\begin{aligned} X_t^{(T)} &:= \mathbf{E}_{\mathbf{Q}}[X_T \mid \mathcal{F}_t] \\ &= x + \frac{1}{\lambda}H_T(\mathbf{Q}|\mathbf{P}) - \frac{1}{\lambda\sigma}\int_0^t \left(\rho Y_s - \frac{1}{2}\sigma^2\right) dW_s^{\mathbf{Q}} - \frac{1}{2\lambda}\int_0^t \frac{1}{\sigma^2}\left(\rho Y_s - \frac{1}{2}\sigma^2\right)^2 ds \\ &\quad - \frac{1}{2\lambda}\mathbf{E}_{\mathbf{Q}}\left[\int_t^T \frac{1}{\sigma^2}\left(\rho Y_s - \frac{1}{2}\sigma^2\right)^2 ds \,\middle|\, \mathcal{F}_t\right]. \end{aligned}$$

The last term can be computed explicitly using the conditional moments

$$\mathbf{E}_{\mathbf{Q}}[Y_s \mid \mathcal{F}_t] = \sigma W_t^{\mathbf{Q}} + y_0 - \frac{1}{2}\sigma^2 s$$

and

$$\mathbf{E}_{\mathbf{Q}}[Y_s^2 \mid \mathcal{F}_t] = \sigma^2(s-t) + \sigma^2\left(W_t^{\mathbf{Q}}\right)^2 + \left(y_0 - \frac{1}{2}\sigma^2 s\right)^2 + 2\sigma W_t^{\mathbf{Q}}\left(y_0 - \frac{1}{2}\sigma^2 s\right).$$

Since the components of bounded variation in our equation for the **Q**-martingale $X_t^{(T)}$ must sum up to the constant x, we finally obtain

$$X_t^{(T)} = x + \frac{1}{\lambda}\int_0^t \left[\frac{1}{\sigma}\left(\frac{1}{2}\sigma^2 - \rho Y_s\right) - \frac{\rho^2}{\sigma}(T-s)Y_s\frac{\sigma}{4} + \frac{\sigma}{4\rho^2}\left(\rho + \rho^2(T-s)\right)^2\right] dW_s^{\mathbf{Q}}$$
$$= x + \frac{1}{\lambda}\int_0^t \frac{1}{S_s}\left[\left(-\frac{\rho}{\sigma^2} - \frac{\rho^2}{\sigma^2}(T-s)\right)Y_s + \frac{1}{4}(1+\rho(T-s))^2 + \frac{1}{4}\right] dS_s.$$

This shows that the integrand $(\xi_t^{(T)})$ defined via (24) is indeed given by (35). □

5.3 Power utility

Consider the power utility function $U(x) = \frac{1}{\alpha}x^\alpha$ with parameter $\alpha \in\,]-\infty, 1[\, \setminus \{0\}$. Since $(U')^{-1}(y) = y^\gamma$ with $\gamma := \frac{1}{\alpha-1} \in\,]-\infty, 0[$, the optimal contingent claim for $T > 0$ is given by

$$X_T = x\, Z_T^\gamma\, \mathbf{E_Q}\left[Z_T^\gamma\right]^{-1} = x\, Z_T^\gamma\, \mathbf{E_P}\left[Z_T^\beta\right]^{-1}, \tag{36}$$

and the maximal expected utility

$$u_T(x) = \mathbf{E_P}[U(X_T)] \tag{37}$$

takes the form

$$u_T(x) = \frac{x^\alpha}{\alpha}\mathbf{E_P}\left[Z_T^\beta\right]^{1-\alpha} \tag{38}$$

where

$$\beta := \frac{\alpha}{\alpha - 1} \in\,]-\infty, 1[\, \setminus \{0\}.$$

The following proposition provides an explicit formula for $u_T(x)$. In particular it allows us to compute its rate of growth as $T \nearrow \infty$. Note that for $\alpha > 0$ Eq. (40) describes the exponential growth of $u_T(x)$ to infinity, while for $\alpha < 0$ it specifies the exponential decay of the distance between $u_T(x)$ and its maximal value 0.

Proposition 5.6 *We have*

$$\mathbf{E_P}[Z_T^\beta] = (A_T^-)^{-\frac{1}{2}} \exp\left(B_T + (A_T^-)^{-1} C_T\right), \tag{39}$$

where

$$A_T^\pm := 1 - \frac{1}{2}\left(1 - \sqrt{1-\beta}\right)\left(1 \pm \exp\left(-2\rho\sqrt{1-\beta}\, T\right)\right),$$
$$B_T := \frac{1}{2}\beta y_0 - \frac{1}{2}\frac{\rho}{\sigma^2}\left(\sqrt{1-\beta} - (1-\beta)\right) y_0^2 - \left[\frac{1}{8}\beta\sigma^2 + \frac{\rho}{2}\left(\sqrt{1-\beta} - (1-\beta)\right)\right] T,$$

and

$$C_T := -\frac{1}{2} y_0 \beta \exp\left(-\rho\sqrt{1-\beta}T\right) + \frac{1}{2} y_0^2 \frac{\rho}{\sigma^2}\left(\sqrt{1-\beta} - (1-\beta)\right)\exp\left(-2\rho\sqrt{1-\beta}\, T\right)$$
$$+ \frac{1}{16}\frac{\beta^2\sigma^2}{\rho}(1-\beta)^{-\frac{1}{2}}\left(1 - \exp\left(-2\rho\sqrt{1-\beta}\, T\right)\right).$$

In particular, the maximal expected utility grows at the rate

$$\lim_{T\nearrow\infty}\frac{1}{T}\log\left(|u_T(x)|\right)=\frac{1}{8}\alpha\sigma^2+\frac{1}{2}\rho(1-\sqrt{1-\alpha}). \tag{40}$$

Proof In order to compute the expectation of

$$Z_T^\beta=\exp\left(\frac{\rho\beta}{\sigma^2}\int_0^T Y_s dY_s+\frac{1}{2}\frac{\beta\rho^2}{\sigma^2}\int_0^T Y_s^2 ds-\frac{1}{2}\beta(Y_T-y_0)-\frac{1}{8}\beta\sigma^2 T\right),$$

we first eliminate the energy term $\int_0^T Y_s^2 ds$ appearing in the exponent by means of a suitable Girsanov transformation. For $\delta>0$ we denote by $\mathbf{P}^\delta$ the probability measure on $\mathcal{F}_T$ with density

$$\begin{aligned}\varphi_T^\delta &:= \exp\left(\int_0^T\frac{\rho-\delta}{\sigma}Y_s dW_s-\frac{1}{2}\int_0^T\left(\frac{\rho-\delta}{\sigma}Y_s\right)^2 ds\right)\\ &= \exp\left(\frac{\rho-\delta}{\sigma^2}\int_0^T Y_s dY_s+\frac{1}{2}\frac{\rho^2-\delta^2}{\sigma^2}\int_0^T Y_s^2 ds\right)\end{aligned} \tag{41}$$

with respect to $\mathbf{P}$. Since

$$W_t^\delta := W_t-\int_0^t\frac{\rho-\delta}{\sigma}Y_s ds$$

defines a Wiener process under $\mathbf{P}^\delta$, $(Y_t)_{0\le t\le T}$ becomes an Ornstein–Uhlenbeck process under $\mathbf{P}^\delta$ with parameter δ, i.e.,

$$dY_t=-\delta Y_t dt+\sigma dW_t^\delta. \tag{42}$$

Setting $\delta:=\rho\sqrt{1-\beta}$ and using Itô's formula

$$\int_0^T Y_s dY_s=\frac{1}{2}(Y_T^2-y_0^2)-\frac{1}{2}\sigma^2 T,$$

we obtain

$$\begin{aligned}\mathbf{E}_{\mathbf{P}}[Z_T^\beta] &= \mathbf{E}^\delta\left[Z_T^\beta(\varphi_T^\delta)^{-1}\right]\\ &= \mathbf{E}^\delta\left[\exp\left(\frac{\rho}{\sigma^2}\left(\sqrt{1-\beta}-(1-\beta)\right)\int_0^T Y_s dY_s-\frac{1}{2}\beta(Y_T-y_0)-\frac{1}{8}\beta\sigma^2 T\right)\right]\\ &= \exp(B_T)\,\mathbf{E}^\delta\left[\exp\left(\frac{\rho}{2\sigma^2}\left(\sqrt{1-\beta}-(1-\beta)\right)Y_T^2-\frac{1}{2}\beta Y_T\right)\right].\end{aligned}$$

In view of (42), Y_T is Gaussian with mean $m:=e^{-\delta T}y_0$ and variance $\nu^2:=\frac{\sigma^2}{2\delta}(1-e^{-2\delta T})$. Using the fact that

$$\mathbf{E}[\exp(\lambda Y^2+\eta Y)]=(1-2\lambda\nu^2)^{-\frac{1}{2}}\exp\left((1-2\lambda\nu^2)^{-1}\left(\lambda m^2+\eta m+\frac{1}{2}\eta^2\nu^2\right)\right) \tag{43}$$

for any random variable Y with normal distribution $N(m, \nu^2)$ and for $\lambda \nu^2 < \frac{1}{2}$, we finally obtain Eq. (39). Combining (38) with (39), we see that

$$\log |u_T(x)| = \alpha \log x - \log |\alpha| + (1-\alpha)\left(B_T + (A_T^-)^{-1} C_T\right) - \frac{1}{2}(1-\alpha) \log A_T^-.$$

Since A_T^- and C_T converge to a finite limit as $T \nearrow \infty$,

$$\begin{aligned} \lim_{T \nearrow \infty} \frac{1}{T} \log |u_T(x)| &= (1-\alpha) \lim_{T \nearrow \infty} \frac{1}{T} B_T \\ &= (\alpha - 1)\left[\frac{1}{8}\beta\sigma^2 + \frac{\rho}{2}\left(\sqrt{1-\beta} - (1-\beta)\right)\right] \\ &= \frac{1}{8}\alpha\sigma^2 + \frac{1}{2}\rho\left(1 - \sqrt{1-\alpha}\right). \end{aligned}$$

□

Our next goal is to identify the optimal strategy $(\xi_t^{(T)})_{0 \le t \le T}$ defined by (24). It will be described in terms of the optimal proportion $\pi_t^{(T)}$ of the capital

$$X_t^{(T)} = \mathbf{E_Q}[X_T | \mathcal{F}_t]$$

which should be invested in the financial asset at time t for any $t \in [0, T]$, see (25).

Proposition 5.7 *The optimal proportion $\pi_t^{(T)}$ is an affine function of the logarithmic stock price given by*

$$\pi_t^{(T)} = a(T-t) Y_t + b(T-t), \tag{44}$$

where

$$a(T-t) := -\frac{\rho}{\sigma^2}\sqrt{1-\beta}\, A_{T-t}^+ (A_{T-t}^-)^{-1}$$

and

$$b(T-t) := \frac{1}{2}\left[1 - (A_{T-t}^-)^{-1} \beta \exp\left(-\rho\sqrt{1-\beta}(T-t)\right)\right].$$

In particular, the asymptotic form of the optimal strategy for $T \nearrow \infty$ is given by

$$\pi_t := \lim_{T \nearrow \infty} \pi_t^{(T)} = -\frac{\rho}{\sigma^2 \sqrt{1-\alpha}} Y_t + \frac{1}{2}. \tag{45}$$

Proof Consider the $\mathbf{Q}$-martingale $(M_t)_{0 \le t \le T}$ defined by

$$M_t := \mathbf{E_Q}[Z_T^\gamma | \mathcal{F}_t],$$

and recall the measure $\mathbf{P}^\delta$ introduced in the proof of Proposition 5.6 for $\delta := \rho\sqrt{1-\beta}$. In terms of its densities

$$\varphi_t^\delta := \mathbf{E_P}[\varphi_T^\delta | \mathcal{F}_t]$$

with respect to $\mathbf{P}$ where φ_T^δ is given by (41), M_t takes the form

$$\begin{aligned} M_t &= Z_t^{-1} \mathbf{E_P}\left[Z_T^\beta | \mathcal{F}_t\right] \\ &= Z_t^{-1} \varphi_t^\delta \mathbf{E}^\delta \left[Z_T^\beta (\varphi_T^\delta)^{-1} | \mathcal{F}_t\right] \\ &= L_t\, \mathbf{E}^\delta \left[\exp\left(\frac{1}{2}\frac{\rho}{\sigma^2}\left(\sqrt{1-\beta} - (1-\beta)\right) Y_T^2 - \frac{1}{2}\beta Y_T\right) \middle| \mathcal{F}_t\right] \end{aligned}$$

with

$$\log L_t := -\frac{\delta}{\sigma^2}\int_0^t Y_s dY_s - \frac{1}{2}\frac{\delta^2}{\sigma^2}\int_0^t Y_s^2 ds + \frac{1}{2}(Y_t - y_0) + \frac{1}{8}\sigma^2 t + B_T$$

where B_T was defined in Proposition 5.6.

In view of (42) the random variable Y_T is Gaussian under the conditional distribution $\mathbf{P}^\delta[\,.\,|\mathcal{F}_t]$, with conditional mean $m := Y_t \exp(-\delta(T-t))$ and conditional variance $\rho^2 := \frac{\sigma^2}{2\delta}(1-\exp(-2\delta(T-t)))$. Using again formula (43) we finally obtain an expression of the form

$$M_t = \exp(N_t)D_t,$$

where

$$N_t := \int_0^t \pi_t^{(T)}\sigma dW_s^{\mathbf{Q}}$$

is a $\mathbf{Q}$-martingale, $\pi_t^{(T)}$ is given by (44) and $(D_t)_{0\le t\le T}$ is some adapted process with continuous paths of bounded variation. But $(M_t)_{0\le t\le T}$ is a $\mathbf{Q}$-martingale, and this implies

$$M_t = M_0 \exp\left(N_t - \frac{1}{2}\langle N\rangle_t\right),$$

hence

$$dM_t = M_t dN_t$$

and

$$\begin{aligned} dX_t^{(T)} &= X_t^{(T)} dN_t \\ &= X_t^{(T)}\pi_t^{(T)}\sigma dW_t^{\mathbf{Q}} \\ &= X_t^{(T)}\pi_t^{(T)} S_t^{-1} dS_t. \end{aligned}$$

Thus we have shown that the trading strategy $(\xi_t^{(T)})$ in (24) is given by

$$\xi_t^{(T)} = X_t^{(T)}\pi_t^{(T)} S_t^{-1},$$

and so the quantity $\pi_t^{(T)}$ defined by (44) is indeed the optimal proportion of the available capital $X_t^{(T)}$ which should be invested in the financial asset at time t.

Since

$$\lim_{T\nearrow\infty} A_T^{\pm} = 1 - \frac{1}{2}\left(1 - \sqrt{1-\beta}\right),$$

we obtain

$$\lim_{T\nearrow\infty} a(T-t) = -\frac{\rho}{\sigma^2}\sqrt{1-\beta} = -\frac{\rho}{\sigma^2\sqrt{1-\alpha}}$$

and

$$\lim_{T\nearrow\infty} b(T-t) = \frac{1}{2},$$

and so the asymptotic form of the strategy for $T \nearrow \infty$ is given by (45). □

Remark 5.8 In Fleming and Sheu [13] dynamic programming methods are used in order to compute directly the optimal growth rate

$$\Lambda = \sup \limsup_{T \nearrow \infty} \frac{1}{T} \log \mathbf{E_P}[U(X_T^\pi)]$$

for power utilities U, where the supremum is taken over all admissible trading strategies; see also Pham [28]. Our Propositions 5.6 and 5.7 provide, in addition, explicit results for any finite horizon T and may be viewed as a probabilistic complement to the analytical method in [13].

Remark 5.9 We have considered prices and contingent claims in discounted form since the bond was assumed to be normalized to $B_t \equiv 1$. For a constant risk-free interest rate $r > 0$, the undiscounted contingent claim generated by a self-financing trading strategy is of the form $\widetilde{X}_T = X_T e^{rT}$, and it seems natural to apply a given utility function U to $\widetilde{X}_T$ rather than to X_T. Moreover one may want to introduce a subjective rate of discounting $\delta > 0$. Let us denoty by $\widetilde{u}_T(x)$ the optimal value obtained by maximizing

$$\mathbf{E}\left[U\left(\widetilde{X}_T\right)\right] e^{-\delta T}.$$

For a power utility $U(x) = \frac{1}{\alpha}x^\alpha$ the optimal contingent claim is clearly the same as before, i.e., $\widetilde{X}_T = X_T e^{rT}$ where X_T is given by (36). Note, however, that Z_T now depends on r. The optimal value takes the form

$$\widetilde{u}_T(x) = u_T(x) e^{(\alpha r - \delta)T}$$

where $u_T(x)$ is given by (37), and in analogy to Proposition 5.6 we obtain exponential growth at the rate

$$\lim_{T \nearrow \infty} \frac{1}{T} \log(|\widetilde{u}_T(x)|) = \alpha \left(r + \frac{1}{2\sigma^2} \left(\frac{1}{2}\sigma^2 - r \right)^2 \right) + \frac{1}{2}\rho\left(1 - \sqrt{1-\alpha}\right) - \delta;$$

see [23] for detailed computations and for extensions to more general Ornstein–Uhlenbeck processes in a robust setting.

5.4 Certainty equivalents and their growth rates

We now shall interpret in financial terms the previous results on optimal investment with respect to the geometric Ornstein–Uhlenbeck process for the utility functions considered above. We shall also analyze to which extent the logarithmic utility $U(x) = \log(x)$ and the exponential utility $U(x) = -\exp(-x)$ correspond to the limiting cases $\alpha \to 0$ and $\alpha \to -\infty$ for power utilities $U^{(\alpha)}(x) = \frac{x^\alpha}{\alpha}$.

To make the above results comparable we transform them from the utility scale to the money scale, using the concept of "*certainty equivalent*", due to De Finetti (compare, e.g., [7, Example 3.3.5]).

We fix an initial endowment x of an economic agent, which we shall eventually normalize by $x = 1$, as well as the value function $u_T(x)$. The certainty equivalent $\mathrm{CE}_T(x)$ then is the solution to

$$U(x + \mathrm{CE}_T(x)) = u_T(x). \tag{46}$$

The interpretation of this formula is that an agent whose preferences are modeled by expected utility U at time T is indifferent between having an initial endowment $x + \mathrm{CE}_T(x)$

without the possibility of investing in the financial market S (so that her wealth remains constant during $[0, T]$), as compared to having an initial endowment x as well as the possibility of investing (optimally) in the market S during $[0, T]$.

By scaling we have $\mathrm{CE}_T(x) = x\mathrm{CE}_T(1)$ in the case of logarithmic and power utility while $\mathrm{CE}_T(x)$ is independent of x in the case of the exponential utility. We therefore shall simply write CE_T for $\mathrm{CE}_T(1)$; if we want to emphasize the role of the utility function U, we shall also denote this quantity by CE_T^U.

Propositions 5.1, 5.4 and 5.6 yield an explicit description of the growth of the certainty equivalents as $T \nearrow \infty$.

5.4.1 Logarithmic utility

We have

$$\mathrm{CE}_T^{(\log)}(x) = x\left(\exp H_T(\mathbf{P}|\mathbf{Q}) - 1\right),$$

and formula (29) yields exponential growth at the rate

$$\lim_{T\nearrow\infty} \frac{1}{T} \log \mathrm{CE}_T^{(\log)}(x) = \frac{\rho}{4} + \frac{\sigma^2}{8}. \tag{47}$$

5.4.2 Exponential utility

We have

$$\mathrm{CE}_T^{(\exp)}(x) = \frac{1}{\lambda} H_T(\mathbf{Q}|\mathbf{P}),$$

and formula (34) yields cubic growth at the rate

$$\lim_{T\nearrow\infty} \frac{1}{T^3} \mathrm{CE}_T^{(\exp)}(x) = \frac{1}{\lambda}\frac{\rho^2\sigma^2}{24}. \tag{48}$$

5.4.3 Power utility

For $\alpha \in\,]-\infty, 1[\, \setminus \{0\}$ and $U(x) = \frac{x^\alpha}{\alpha}$, the certainty equivalent is given by

$$\mathrm{CE}_T^{(\alpha)}(x) = x\left(\mathbf{E}_{\mathbf{P}}\left[Z_T^{\beta}\right]^{-\frac{1}{\beta}} - 1\right),$$

and formula (40) yields exponential growth at the rate

$$\lim_{T\nearrow\infty} \frac{1}{T} \log \mathrm{CE}_T^{(\alpha)}(x) = \frac{\sigma^2}{8} + \frac{\rho(1-\sqrt{1-\alpha})}{2\alpha}. \tag{49}$$

Remark 5.10 For the optimization problem formulated in Remark 5.9, the corresponding certainty equivalent $\widetilde{CE}_T(x)$ is given by

$$U\left(x + \widetilde{CE}_T(x)\right) = \widetilde{u}_T(x).$$

For a power utility $U(x) = \frac{1}{\alpha}x^\alpha$ we obtain

$$x + \widetilde{CE}_T^{(\alpha)}(x) = \left(x + CE_T^{(\alpha)}(x)\right) e^{\left(r-\frac{\delta}{\alpha}\right)T},$$

and (49) implies exponential growth at the rate

$$\lim_{t\nearrow\infty}\frac{1}{T}\log\widetilde{CE}_T^{(\alpha)}(x)=\frac{1}{2\sigma^2}\left(\frac{\sigma^2}{2}-r\right)^2+\frac{\rho(1-\sqrt{1-\alpha})}{2\alpha}+r-\frac{\delta}{\alpha}.$$

We now discuss the limiting behavior $\alpha\to 0$ and $\alpha\to-\infty$. We start with the first case which is easier.

Noting that $\lim_{\alpha\to 0}\frac{x^\alpha-1}{\alpha}=\log x$ and $\lim_{\alpha\to 0}\frac{1-\sqrt{1-\alpha}}{2\alpha}=\frac{1}{4}$, we have that (49) tends to (47) as $\alpha\to 0$. In fact, a stronger result holds true: one may verify using (28), (38), (39) that, for fixed T, we have

$$\lim_{\alpha\to 0}\mathrm{CE}_T^{(\alpha)}=\mathrm{CE}_T^{(\log)}.$$

We thus see that

$$\lim_{\alpha\to 0}\lim_{T\to\infty}\frac{1}{T}\log\left(\mathrm{CE}_T^{(\alpha)}\right)=\lim_{T\to\infty}\frac{1}{T}\log\left(\mathrm{CE}_T^{(\log)}\right)=\lim_{T\to\infty}\lim_{\alpha\to 0}\frac{1}{T}\log\left(\mathrm{CE}_T^{(\alpha)}\right)$$

This formula indicates that the case of logarithmic utility indeed corresponds to the limit $\alpha\to 0$ for power utility.

We observe that the growth rate $\lim_{T\to\infty}\frac{1}{T}\log\left(\mathrm{CE}_T^{(\alpha)}\right)$ of the certainty equivalent is a monotone function in $\alpha\in\,]-\infty,1[$ ranging from

$$\lim_{\alpha\to-\infty}\lim_{T\to\infty}\frac{1}{T}\log\left(\mathrm{CE}_T^{(\alpha)}\right)=\frac{\sigma^2}{8}$$

via

$$\lim_{\alpha\to 0}\lim_{T\to\infty}\frac{1}{T}\log\left(\mathrm{CE}_T^{(\alpha)}\right)=\lim_{T\to\infty}\frac{1}{T}\log\left(\mathrm{CE}_T^{(\log)}\right)=\frac{\sigma^2}{8}+\frac{\rho}{4}$$

to

$$\lim_{\alpha\to 1}\lim_{T\to\infty}\frac{1}{T}\log\left(\mathrm{CE}_T^{(\alpha)}\right)=\frac{\sigma^2}{8}+\frac{\rho}{2}.$$

This reflects the financial intuition that an agent with smaller risk aversion can take better advantage (measured in terms of certainty equivalents) of investment opportunities.

The analysis of the behavior for $\alpha\to-\infty$ is more subtle than the case $\alpha\to 0$. Recall that the exponential utility represents the limit of $U^{(\alpha)}(x)=\frac{x^\alpha}{\alpha}$, as $\alpha\to-\infty$, after proper affine normalisation:

$$-\exp(-x)=-\lim_{\alpha\to-\infty}\left(1-\frac{x}{\alpha}\right)^\alpha=\lim_{\alpha\to-\infty}\frac{(x-\alpha)^\alpha}{\alpha}|\alpha|^{-\alpha+1}.$$

Hence, up to the multiplicative factor $|\alpha|^{-\alpha+1}$, which is irrelevant for the certainty equivalent, the exponential utility $-\exp(-x)$ is close to the shifted power utility $U^{(\alpha)}(x-\alpha)$, as $\alpha\to-\infty$.

Using (34), (38), (39) one may verify that *for fixed horizon* T, we again have

$$\lim_{\alpha\to-\infty}\mathrm{CE}_T^{(\alpha)}(|\alpha|)=\mathrm{CE}_T^{(\exp)}. \tag{50}$$

However, the above relation does not carry over to the limiting expressions for $T\to\infty$ by interchanging the limiting procedures for α and T as in the logarithmic case above: the terms in (48) and (49) are of a completely different qualitative structure. In fact, the term T^3 in (48) looks at first sight rather puzzling.

In order to develop an understanding of the situation it is instructive to have a closer look at the *optimal trading strategies*. Let us again start with the logarithmic case which is the easiest, as $\pi_t^{(\log,T)} = \frac{1}{2} - \frac{1}{\sigma^2}\rho Y_t$, as defined in (31), does not depend on T, so that we may define $\pi_t^{(\log)} = \frac{1}{2} - \frac{1}{\sigma^2}\rho Y_t$ independently of the horizon T.

As regards the optimal trading strategy $\pi_t^{(\alpha,T)}$ for power utility (44), it does depend on the horizon T; nevertheless the limit $\pi_t^{(\alpha)}$ for $T \to \infty$ exists for each $\alpha \in \,]-\infty, 1[\setminus\{0\}$ (45) and one readily verifies that

$$\pi_t^{(\log)} = -\frac{\rho}{\sigma^2} Y_t + \frac{1}{2} = \lim_{\alpha\to 0} -\frac{\rho}{\sigma^2\sqrt{1-\alpha}} Y_t + \frac{1}{2} = \lim_{\alpha\to 0} \pi_t^{(\alpha)},$$

in order words that the logarithmically optimal investment strategy again is the limit of the limiting α-optimal strategy $\pi^{(\alpha)}$, as $\alpha \to 0$.

Passing to the exponential utility, the picture changes drastically: the optimal quantity $\xi_t^{(\exp,T)}$, given by (35), depends on the horizon T in such a way that it is not possible to pass to the limit $T \to \infty$ as the leading term is $\frac{\rho^2}{4\lambda S_t}(T-t)^2$.

In order to find the reason why the exponential utility maximizer shows this rather strange behavior let us rewrite the value functions (27), (33) and (38) in a form appropriate to argue with the Hamilton–Jacobi-Bellman equation.

$$\begin{aligned} u_T^{(\log)}(x,y,t) &= \log x + \frac{1}{4}\rho(T-t) - \frac{1}{8}\left(1-e^{-2\rho(T-t)}\right) + \frac{1}{4}\frac{\rho}{\sigma^2}y^2\left(1-e^{-2\rho(T-t)}\right) \\ &\quad -\frac{1}{2}y\left(1-e^{-\rho(T-t)}\right) + \frac{1}{8}\sigma^2(T-t), \end{aligned} \tag{51}$$

$$\begin{aligned} u_T^{(\exp)}(x,y,t) &= -\exp\left[-\lambda x - \left(\frac{1}{2}\frac{\rho^2}{\sigma^2}y^2 - \frac{\rho}{2}y + \frac{1}{8}\sigma^2\right)(T-t)\right. \\ &\quad \left. -\left(\frac{1}{4}\rho^2(1-y) + \frac{1}{8}\rho\sigma^2\right)(T-t)^2 - \frac{1}{24}\rho^2\sigma^2(T-t)^3\right], \end{aligned} \tag{52}$$

$$u_T^{(\alpha)}(x,y,t) = \frac{x^\alpha}{\alpha}\left[\left(A_{T-t}^-\right)^{-\frac{1}{2}} \exp\left(B_{T-t} + \left(A_{T-t}^-\right)^{-1} C_{T-t}\right)\right]^{1-\alpha}, \tag{53}$$

where

$$\begin{aligned} A_{T-t}^{\pm} &:= 1 - \frac{1}{2}\left(1-\sqrt{1-\beta}\right)\left(1 \pm \exp\left(-2\rho\sqrt{1-\beta}\,(T-t)\right)\right), \\ B_{T-t} &:= \frac{1}{2}\beta y - \frac{1}{2}\frac{\rho}{\sigma^2}\left(\sqrt{1-\beta} - (1-\beta)\right)y^2 \\ &\quad -\left[\frac{1}{8}\beta\sigma^2 + \frac{\rho}{2}\left(\sqrt{1-\beta} - (1-\beta)\right)\right](T-t), \text{ and} \\ C_{T-t} &:= -\frac{1}{2}y\beta\exp\left(-\rho\sqrt{1-\beta}(T-t)\right) \\ &\quad +\frac{1}{2}y^2\frac{\rho}{\sigma^2}\left(\sqrt{1-\beta} - (1-\beta)\right)\exp\left(-2\rho\sqrt{1-\beta}(T-t)\right) \\ &\quad +\frac{1}{16}\frac{\beta^2\sigma^2}{\rho}(1-\beta)^{-\frac{1}{2}}\left(1-\exp\left(-2\rho\sqrt{1-\beta}(T-t)\right)\right). \end{aligned}$$

A basic feature of dynamic programming is that, fixing T and plugging the optimizer $\left(\widehat{X}_t^{(T)}\right)_{0\le t\le T}$ as well as the process $(Y_t)_{0\le t\le T}$ into the value function above, one obtains a (local) martingale $\left(u_T\left(\widehat{X}_t^{(T)}, Y_t, t\right)\right)_{0\le t\le T}$. We do not try to prove this rigorously (which is possible) as we only want to argue formally to develop an intuition for the phenomena encountered above. We shall also use the formal identities $\mathbf{E}[dW_t] = 0$ and $(dW_t)^2 = dt$.

We obtain from the martingale property the equation

$$\mathbf{E}\left[d\left(u_T\left(\widehat{X}_t^{(T)}, Y_t, t\right)\right)\right] = 0, \tag{54}$$

where Itô's formula allows us to write

$$\begin{aligned}\mathbf{E}\left[d\left(u_T\left(\widehat{X}_t^{(T)}, Y_t, t\right)\right)\right] &= \frac{\partial u_T}{\partial t}dt + \frac{\partial u_T}{\partial x}\mathbf{E}\left[d\widehat{X}_t^{(T)}\right] + \frac{\partial u_T}{\partial y}\mathbf{E}\left[dY_t\right] + \frac{\partial^2 u_T}{2\partial x^2}\left(d\widehat{X}_t^{(T)}\right)^2 \\ &\quad + \frac{\partial^2 u_T}{2\partial y^2}\left(dY_t\right)^2 + \frac{\partial^2 u_T}{\partial x \partial y} d\widehat{X}_t^{(T)} dY_t \end{aligned} \tag{55}$$

Analyzing the above equation, it turns out that the mixed derivative $\frac{\partial^2 u_T}{\partial x \partial y}(x, y, t)$ plays a crucial role: while this term vanishes in the case of logarithmic utility (51), it cannot be neglected in the exponential (52) and the power case (53).

Concentrating on the exponential case we deduce from (52) that the leading term (w.r. to $T - t$) of $\frac{\partial^2 u_T^{(\exp)}}{\partial x \partial y}$ is given by

$$\frac{\partial^2 u_T^{(\exp)}}{\partial x \partial y} \approx -\frac{\lambda \rho^2}{4}(T-t)^2 u_T^{(\exp)}.$$

This term dominates (for large $T-t$) the derivatives $\frac{\partial u_T^{(\exp)}}{\partial x}$ as well as $\frac{\partial^2 u_T^{(\exp)}}{\partial x^2}$, which equal $-\lambda u_T^{(\exp)}$ and $\lambda^2 u_T^{(\exp)}$ respectively.

This gives us a clue to the understanding of the leading term of the optimal strategy

$$\xi_t^{(T)} \approx \frac{\rho^2}{4\lambda S_t}(T-t)^2. \tag{56}$$

We may interpret the dynamic programming equation (54) in the following way: the exponential utility maximizer chooses at time t the investment $\xi_t^{(T)}$ in such a way that, when plugging $d\widehat{X}_t^{(T)} = \xi_t^{(T)} dS_t$ into (54) and (55) the term $\frac{\partial u_T}{\partial t}$ becomes minimal. Indeed, in this case the function $u_T(.,.,0)$ becomes maximal for the given boundary condition $u_T(x, y, T) = -\exp(-\lambda x)$ as the descent in the time coordinate is steepest. The choice of the control variable $\xi_t^{(T)}$ in $d\widehat{X}_t^{(T)} = \xi_t^{(T)} dS_t$ does not effect the behavior of $\mathbf{E}[dY_t]$ or $(dY_t)^2$; hence, we see from (55) that one has to solve the following maximization problem for the variable $\xi \in \mathbb{R}$:

$$\frac{\partial u_T}{\partial x}\xi \mathbf{E}[dS_t] + \frac{\partial^2 u_T}{2\partial x^2}\xi^2 (dS_t)^2 + \frac{\partial^2 u_T}{\partial x \partial y}\xi dS_t dY_t \longrightarrow \max!$$

Using

$$\frac{\partial u_T}{\partial x} = -\lambda u_T, \quad \frac{\partial^2 u_T}{\partial x^2} = \lambda^2 u_T, \quad \frac{\partial^2 u_T}{\partial x \partial y} \approx -\frac{\lambda \rho^2}{4}(T-t)^2 u_T$$

and

$$\mathbf{E}[dS_t] = e^y\left(\frac{\sigma^2}{2} - \rho y\right)dt, \quad (dS_t)^2 = e^{2y}\sigma^2 dt, \quad dS_t dY_t = \sigma^2 e^y dt$$

we arrive, letting $S_t = e^y$ and keeping only the leading terms in $(T-t)^2$, at

$$\frac{\lambda^2}{2} u_T \xi^2 e^{2y} \sigma^2 dt - \frac{\lambda \rho^2}{4} (T-t)^2 u_T \xi \sigma^2 e^y dt \longrightarrow \max!$$

Differentiating with respect to ξ and equating to zero yields (56).

Summing up, we see that in the exponential case the "joint wobbling" $d\widehat{X}_t^{(T)} dY_t$ and the mixed derivative $\frac{\partial^2 u_T}{\partial x \partial y}$ are the crucial terms which cause the optimal value $\xi_t^{(T)} S_t = \frac{\rho^2}{4\lambda}(T-t)^2$ to be of the order $(T-t)^2$, independently of the current value of the process Y_t which determines the market price of risk.

Now we are in a position to understand why the double limiting procedure $\alpha \to -\infty$, $T \to \infty$ cannot be interchanged: For fixed horizon T we have that $u_T^{(\alpha)}(x-\alpha)$ tends to $u_T^{(\exp)}(x)$, as $\alpha \to -\infty$ for $x \in \mathbb{R}$, and the same limiting behavior holds true for the corresponding trading strategies. On the other hand the optimal exponential strategies $\xi_t^{(T)}$ (56) do not converge, as $T \to \infty$, while we have seen that, for fixed $\alpha \in [-\infty, 1[\setminus\{0\}$ the optimal power strategies $\pi_t^{(\alpha,T)}$ do converge to $\pi_t^{(\alpha)}$.

Not only in the case of exponential utility we find remarkable differences to the Black–Scholes situation where the optimal strategies are independent of the time horizon, but even in the case of logarithmic utility, which is the most regular one, we find unexpected phenomena. As regards the optimal investment of the logarithmic utility maximizer there are no surprises yet: the optimal investment proportion $\pi_t^{(\log)}$ given in (31) is proportional to the market price of risk $-\frac{\rho}{\sigma} Y_t + \frac{\sigma}{2}$ of the process S (see (14)). When $Y_t = \frac{\sigma^2}{2\rho}$ there are no profitable investment opportunities as in this case the market price of risk vanishes: the optimal strategy of the log-optimizer therefore is not to invest into the risky asset in this situation.

Now let us look at the value function (51) for the logarithmic utility

$$\begin{aligned} u_T^{(\log)}(x,y,t) = \log x + \frac{1}{4}\rho(T-t) - \frac{1}{8}\left(1 - e^{-2\rho(T-t)}\right) \\ + \frac{1}{4}\frac{\rho}{\sigma^2} y^2 \left(1 - e^{-2\rho(T-t)}\right) - \frac{1}{2} y \left(1 - e^{-\rho(T-t)}\right) + \frac{1}{8}\sigma^2 (T-t). \end{aligned}$$

Fixing x and t we observe that the minimal value is attained at $y_{\min} \approx \frac{\sigma^2}{\rho}$ for large T. Hence the situation for the log-investor is worse (if measured by expected utility at time T) if $Y_t = \frac{\sigma^2}{\rho}$ than in the case $Y_t = \frac{\sigma^2}{2\rho}$, when the market price of risk vanishes. To explain this at first glance counter-intuitive phenomenon we quote from Wilhelm Tell: "He has no choice but through this sunken way to come to Kussnacht. There is no other road." The drift of the Ornstein–Uhlenbeck process Y drives the process back towards $Y = 0$; hence a typical path of Y, for which at time t we have $Y_t = \frac{\sigma^2}{\rho}$, will tend towards 0 and on this way it has to pass through the region around $\frac{\sigma^2}{2\rho}$ where the log-investor will not find profitable investment opportunities. This situation is worse than starting at $Y_t = \frac{\sigma^2}{2\rho}$, where one already may hope to drift out of the "sunken way".

Let us finally look at the pathwise growth of the capital X_T generated by the optimal strategy up to time T. Note first that

$$\lim_{T \nearrow \infty} \frac{1}{T} \log Z_T = -\left(\frac{\rho}{4} + \frac{\sigma^2}{8}\right) \quad \mathbf{P}\text{-a.s.,} \tag{57}$$

due to (19) and (20).

For a *power utility* with parameter $\alpha \in]-\infty, 1[\setminus\{0\}$ we have

$$X_T = x Z_T^{\gamma} \mathbf{E_P}\left[Z_T^{\beta}\right]^{-1}$$

with $\gamma = \frac{1}{\alpha-1}$ and $\beta = \frac{\alpha}{\alpha-1}$. By (38) and (40),

$$\lim_{T\nearrow\infty} \frac{1}{T} \log \mathbf{E_P}\left[Z_T^{\beta}\right] = \frac{1}{1-\alpha}\left(\frac{1}{8}\alpha\sigma^2 + \frac{1}{2}\rho\left(1-\sqrt{1-\alpha}\right)\right).$$

Together with (57) this yields exponential growth of X_T at the rate

$$\begin{aligned}\lim_{T\nearrow\infty} \frac{1}{T} \log X_T &= \frac{1}{1-\alpha}\left(\frac{1}{8}\sigma^2 + \frac{1}{4}\rho - \frac{1}{8}\alpha\sigma^2 - \frac{1}{2}\rho\big(1-\sqrt{1-\alpha}\big)\right)\\ &= \frac{1}{8}\sigma^2 + \frac{1}{4}\rho\; h(\alpha) \quad \mathbf{P}\text{-almost surely,}\end{aligned} \tag{58}$$

where we put

$$h(\alpha) := \frac{1}{1-\alpha}\big(2\sqrt{1-\alpha}-1\big).$$

Note that $h(\alpha)$ attains its maximal value 1 for $\alpha = 0$. This corresponds to the *logarithmic* case where $X_T = x Z_T^{-1}$, hence

$$\lim_{T\nearrow\infty} \frac{1}{T} \log X_T = \frac{1}{8}\sigma^2 + \frac{1}{4}\rho \quad \mathbf{P}\text{-a.s.,} \tag{59}$$

due to (57). This was to be expected as logarithmic utility optimizes the expected monetary growth rate. Note also that $h(\alpha)$ decreases to 0 as $\alpha \searrow -\infty$. On the other hand, the growth rate becomes negative as soon as $\alpha > \frac{3}{4}$, and it decreases to $-\infty$ as $\alpha \nearrow 1$, thus approaching the risk neutral case.

It seems a remarkable fact that, almost surely for $\alpha \in \left]\frac{3}{4}, 1\right[$, the α-optimal investor eventually looses all her initial endowment x in the long run, at a rate which becomes arbitrarily large as $\alpha \to 1$.

For the *exponential* utility with parameter $\lambda > 0$ we have

$$X_T = x + \frac{1}{\lambda}\left(H_T(\mathbf{Q}|\mathbf{P}) - \log Z_T\right).$$

Combining (57) with (34), we see that X_T grows like T^3 at the rate

$$\lim_{T\nearrow\infty} \frac{1}{T^3} X_T = \frac{1}{\lambda}\frac{\rho^2\sigma^2}{24}.$$

Note that this cubic rate increases to ∞ as λ decreases to 0, which corresponds to the risk neutral case.

6 Cost averaging and exponential growth of wealth

The so-called "cost average effect" is a popular argument among practitioners in finance, in particular among those selling investment plans. We cite from [32]: "cost averaging simply means investing the same fixed amount of money in shares of a risky asset at regular intervals of time. Thus the investor will always buy more shares when prices are low and fewer shares when prices are high. Accordingly the average cost per share is always lower than the average of the share prices over the investment time frame."

The argument is indeed somewhat seducing to convince potential clients to buy investment plans. On the other hand, if one believes that—after proper discounting—the stock price process is a *martingale* [31], then it is obvious that the preceding argument cannot show the superiority of a constant investment over time in comparison to, e.g., a lump sum investment: indeed, it is the basic message of Doob's optional sampling theorem (see, e.g. [29]), that the expected result of any trading strategy (satisfying some regularity condition), when applied to a martingale, simply equals zero. In fact, this argument was already used as "Fundamental Principle" by Bachelier [1].

Of course, the crux with the "cost average" argument sketched in the first paragraph is that, if we try to make it rigorous, one quickly sees that it involves a strategy which fails to be predictable. The two strategies implicitly compared in the above argument are: the strategy A of investing 1 € at each time $t \in \{0, \dots, T-1\}$ into a stock with price process $(S_t)_{t=0}^{T}$ so that one aquires S_t^{-1} stocks at each time t. Eventually one thus holds $\sum_{t=0}^{T-1} S_t^{-1}$ stocks at time T. The alternative is strategy B to invest the total sum of T € in such a way that one buys at each time t the same number x of stocks. Equalling the total investment, this number x is given by $x = \left(\sum_{t=0}^{T-1} S_t/T\right)^{-1}$ so that one eventually holds $T\left(\sum_{t=0}^{T-1} S_t/T\right)^{-1}$ stocks. The latter quantity is indeed always less than or equal than $\sum_{t=0}^{T-1} S_t^{-1}$ as this amounts to comparing the harmonic mean with the arithmetic mean. Hence strategy A indeed dominates strategy B, but strategy B is not predictable as the investment of $x = \left(\sum_{t=0}^{T-1} S_t/T\right)^{-1}$ at each time t involves the knowledge of the process $(S_t)_{t=0}^{T-1}$. Of course, a non-predictable strategy does not make sense economically, so the above argument does not show anything.

On the other hand, if one believes, that the stock price evolution is correctly modeled by a *stationary process*, then the cost averaging effect described above appears as a perfectly sound argument: a stationary process has tendency to be driven back when it is in an unlikely area of its invariant distribution (think, e.g., of the Ornstein–Uhlenbeck process analyzed above); hence the term "lower than the average price," alluded to in the verbal description of the cost average effect above, should make some kind of sense in the framework of stationary processes.

It is a wellknown fact that the notions of martingales and stationary processes are "orthogonal" to each other: the only stationary martingales are the constant processes. On the other hand, from an economic as well as from an intuitive point of view they have much in common: "in the average" they neither move up or down (thinking about the one-dimensional case for simplicity). But, of course, they do so in two completely different ways.

The point of view of modeling with martingales, which goes back to Bachelier [1] and corresponds in modern terminology to the "efficient market hypothesis" in its strong form [31], today is largely considered as too narrow for many applications: for example, it is not possible in this framework to model a different long term yield (in the average) between stocks and bonds. The dominating paradigm in financial modeling today is the class of processes which are *martingales under an equivalent probability measure*. The reason is that—essentially—this is the class of processes which do not allow for arbitrage opportunities. This was the basic insight of the seminal papers by Harrison and Kreps [16], Harrison and Pliska [17] and Kreps [25], and is the theme of the so-called "Fundamental Theorem of Asset Pricing".

For finite time horizon the class of stationary processes and the class of processes which are martingales under an equivalent probability measure are not "orthogonal" any more: for example the Ornstein–Uhlenbeck process clearly admits an equivalent martingale measure for each fixed finite horizon T. The interesting issue is to analyse the asymptotic situation as $T \to \infty$.

In this context the natural class of processes, encompassing as arch-examples the Black–Scholes model as well as the geometric Ornstein–Uhlenbeck process, seems to be those with non-trivial market price of risk process $(\varphi_t)_{t\geq 0}$ in the sense of Definition 1.3, and in particular those where the market price of risk satisfies a large deviations estimate (6). Similar classes of processes have been recently studied by Dempster et al. [9,10,12]. In these papers, it is shown that, under general hypotheses, the strategy of keeping fixed proportions of wealth in the assets under consideration leads to almost sure exponential growth in the value process, as the time horizon T tends to infinity. This fixed proportion strategy bears some similarity with the "cost averaging" approach described above (one is led to buy an asset when its price, relative to the other assets, goes down and sells it when its price goes up again).

Taking up again the sceptical point of view towards cost averaging, there is a long line of research, going back to the work of Constantinides [4] and Dybvig [11], elaborating on the sub-optimality and inefficiency of the cost-averaging investment policy and similar schemes. In the latter paper Ph. Dybvig shows that for an economic agent described as a maximizer of expected utility of terminal wealth for *an arbitrary* utility function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, the final outcome of an optimal portfolio is a random variable which is necessarily anti-comonotone to the Radon–Nikodym derivative $\frac{d\mathbf{Q}}{d\mathbf{P}}$, if $\mathbf{Q}$ is the unique equivalent martingale measure for the (discounted) stock price process. He also shows that, in the Black–Scholes model, the cost-averaging policy (as well as many other popular schemes) simply fails to have this property so that such a policy can be dominated by a better strategy (in the sense of second order stochastic dominance). Compare also [18] on this issue.

Although the above two lines of results seem to go into different directions, they are by no means contradictory. Exponential growth may very well be achieved by sub-optimal strategies such as cost averaging. The crucial issue is the *optimal rate* of exponential growth. The point of the present paper was to carefully analyze these optimal rates, the corresponding strategies, and the relations with asymptotic arbitrage and utility maximization.

Acknowledgments We thank Thomas Knispel for his efficient help in checking and improving the explicit computations for the geometric Ornstein–Uhlenbeck process. We also thank two anonymous referees for their insightful reports which helped us to improve the paper.

References

1. Bachelier L. (1900) Théorie de la Spéculation. Ann. Sci. Ecole Norm. Sup., Vol. 17, pp. 21–86, English translation in: The Random Character of stock prices (P. Costner, editor), MIT Press, 1964
2. Becherer, D.: The numéraire portfolio for unbounded semimartingales. Financ. Stoch. **5**(3), 327–341 (2001)
3. Chamberlain, G., Rothschild, M.: Arbitrage, factor structure, and mean-variance analysis on large asset markets. Econometrica **51**, 1281–1304 (1983)
4. Constantinides, G.: A note on the suboptimality of dollar-cost averaging as an investment policy. J. Financ. Quant. Anal. **14**, 443–450 (1979)
5. Delbaen, F., Schachermayer, W.: A general version of the fundamental theorem of asset pricing. Math. Ann. **300**, 463–520 (1994)
6. Delbaen, F., Schachermayer, W.: The fundamental theorem of asset pricing for unbounded stochastic processes. Math. Ann. **312**, 215–250 (1998)
7. Delbaen, F., Schachermayer, W.: The Mathematics of Arbitrage. Springer, Heidelberg (2006)
8. Dembo, A., Zeitouni, O.: Large Deviations Techniques and Applications. Springer, Heidelberg (1998)
9. Dempster, M., Evstigneev, I., Schenk-Hoppé, K.: Exponential growth of fixed-mix strategies in stationary asset markets. Financ. Stoch. **7**, 263–276 (2003)

10. Dempster, M., Evstigneev, I., Schenk-Hoppé, K.: Volatility-induced financial growth. Research Papers in Management Studies WP 10/2004. Judge Institute of Management, University of Cambridge, October 2004
11. Dybvig, Ph.: Inefficient dynamic portfolio strategies or how to throw away a million dollars in the stock market. Rev. Finan. Stud. **1**, 67–88 (1988)
12. Evstigneev, I., Schenk-Hoppé, K.: From rags to riches: on constant proportions investment strategies. Int. J. Theor. Appl. Financ. **5**, 563–573 (2002)
13. Fleming, W.H., Sheu, S.-J.: Optimal long term growth rate of expected utility of wealth. Ann. Appl. Probab. **9**(3), 871–903 (1999)
14. Florens-Landais, D., Pham, H.: Large deviations in estimation of an Ornstein–Uhlenbeck model. J. Appl. Probab. **36**(1), 60–77 (1999)
15. Föllmer, H., Schweizer, M. : Hedging of Contingent Claims under Incomplete Information. In: Davis, M.H.A., Elliot, R.J. (eds.) Applied Stochastic Analysis, pp. 389–414. Gordon and Breach, New York (1991)
16. Harrison, J.M., Kreps, D.M.: Martingales and arbitrage in multiperiod securities markets. J. Econ. Theory **20**, 381–408 (1979)
17. Harrison, J.M., Pliska, S.R.: Martingales and stochastic integrals in the theory of continuous trading. Stoch. Process. Appl. **11**, 215–260 (1981)
18. Jouini, E.: Arbitrage and investment opportunities. Financ. Stoch. **5**(3), 305–325 (2001)
19. Kabanov, Yu., Kramkov, D.: Asymptotic Arbitrage in large financial markets. Financ. Stoch. **2**, 143–172 (1998)
20. Karatzas, I., Lehoczky, J.P., Shreve, S.E.: Optimal portfolio and consumption decisions for a "small investor" on a finite horizon. SIAM J. Control Optim. **25**, 1557–1586 (1987)
21. Karatzas, I., Lehoczky, J.P., Shreve, S.E., Xu, G.L.: Martingale and duality methods for utility maximization in an incomplete market. SIAM J. Control Optim. **29**, 702–730 (1991)
22. Klein, I., Schachermayer, W.: A quantitative and a dual version of the Halmos-Savage theorem with applications to mathematical finance. Ann. Probab. **24**(2), 867–881 (1996)
23. Knispel, T.: Asymptotics of robust utility maximization. Working Paper HU, Berlin (2008)
24. Kramkov, D., Schachermayer, W.: The condition on the asymptotic elasticity of utility functions and optimal investment in incomplete markets. Ann. Appl. Probab. **9**(3), 904–950 (1999)
25. Kreps, D.M.: Arbitrage and equilibrium in economics with infinitely many commodities. J. Math. Econ. **8**, 15–35 (1981)
26. Merton, R.C.: Lifetime portfolio selection under uncertainty: the continuous case. Rev. Econ. Stat. **51**, 247–257 (1969)
27. Merton, R.C.: Optimum consumption and portfolio rules in a continuous-time model. J. Econ. Theory **8**, 373–413 (1971)
28. Pham, H.: A large deviation approach to optimal long term investment. Financ. Stoch. **7**(2), 169–195 (2003)
29. Rogers, L.C.G., Williams, D.: Diffusions, Markov Processes and Martingales, vol. 1 and 2. Cambridge University Press, Cambridge (2000)
30. Ross, S.A.: The arbitrage theory of asset pricing. J. Econ. Theory **13**(1), 341–360 (1976)
31. Samuelson, P.A.: Proof that properly anticipated prices fluctuate randomly. Ind. Manag. Rev. **6**, 41–50 (1965)
32. Scherer, B., Ebertz, T.: Cost averaging: an expensive strategy for maximizing terminal wealth. Financ. Mark. Portf. Manag. **17**, 186–193 (2003)